LIMIT MEASURES RELATED TO THE CONDITIONALLY FREE CONVOLUTION

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January 2, 2007

Abstract

We describe the limit measures for some class of deformations of the free convolution, introduced by A. D. Krystek and Ł. J. Wojakowski.

Keywords: Cauchy transform; conditionally free product; Jacobi coefficients.

AMS Subject Classification: 46L54, 42C05, 60E10.

1 Introduction

The *conditionally free convolution*, defined by Bożejko, Leinert and Speicher[1], is an associative and commutative operation $\boxplus$ on pairs of compactly supported probability measures on $\mathbb{R}$. It is closely related to the Voiculescu free convolution, namely, if

$$(\mu_1, \nu_1) \boxplus (\mu_2, \nu_2) = (\mu, \nu) \tag{1}$$

then $\nu = \nu_1 \boxplus \nu_2$, and if $\mu_1 = \nu_1$ and $\mu_1 = \nu_2$ then $\mu = \mu_1 \boxplus \mu_2$.

Recall that an important tool for studying a probability measure μ on $\mathbb{R}$ is the *Cauchy transform*, the analytic function $G_\mu : \mathbb{C}_+ \rightarrow \mathbb{C}$ defined by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(t)}{z - t},$$

where $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im} z > 0\}$. If μ is compactly supported then $G_\mu(z)$ can be represented as a continued fraction

$$G_\mu(z) = \frac{1}{z - u_0 - \frac{\alpha_0}{z - u_1 - \frac{\alpha_1}{z - u_2 - \frac{\alpha_2}{z - u_3 - \frac{\alpha_3}{\ddots}}}}},$$

for which we will use more convenient notation:

$$G_\mu(z) = \frac{1}{|z - u_0} - \frac{\alpha_0}{|z - u_1} - \frac{\alpha_1}{|z - u_2} - \frac{\alpha_2}{|z - u_3} - \frac{\alpha_3}{|\dots}. \quad (2)$$

The *Jacobi coefficients* satisfy: $\alpha_k \geq 0$, $u_k \in \mathbb{R}$ and if $\alpha_m = 0$ for some $m \geq 0$ then $\alpha_n = u_n = 0$ for all $n > m$. The coefficient α_0 is called the *variance of μ* and denoted by $V(\mu)$. We will need the following two properties, which can be found in the Chihara monograph[2].

Proposition 1.1 *Assume that μ is a compactly supported probability measure on $\mathbb{R}$, with the Cauchy transform given by (2). Then:*

- (i) *μ is symmetric (i.e. $\mu(A) = \mu(-A)$ holds for every Borel subset of $\mathbb{R}$) if and only if $u_k = 0$ for every $k \geq 0$.*
- (ii) *The support of μ is contained in the halfline $[0, +\infty)$ if and only if there exists a sequence $\{\lambda_m\}_{m \geq 0}$ of nonnegative numbers such that for every $m \geq 0$ we have $\alpha_m = \lambda_{2m} \cdot \lambda_{2m+1}$ and $u_m = \lambda_{2m} + \lambda_{2m-1}$, under convention that $\lambda_{-1} = 0$.*

The numbers λ_m are the nonnegative (i.e. the *upper*) Jacobi coefficients for the symmetric measure μ^{sym} defined by: $\int_{\mathbb{R}} f(t^2) d\mu^{\text{sym}}(t) = \int_{\mathbb{R}} f(t) d\mu(t)$.

To define the conditionally free convolution we define two transforms. For a pair $\mu, \nu \in \mathcal{M}_c$ we define $R_\nu, R_{\mu, \nu}$ as the complex functions which satisfy

$$\frac{1}{G_\nu(z)} = z - R_\nu(G_\nu(z)), \quad (3)$$

$$\frac{1}{G_\mu(z)} = z - R_{\mu, \nu}(G_\nu(z)) \quad (4)$$

(the former is the Voiculescu free transform). For $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{M}_c$ the conditionally free convolution (1) is defined by

$$R_\nu(z) = R_{\nu_1}(z) + R_{\nu_2}(z),$$

$$R_{\mu, \nu}(z) = R_{\mu_1, \nu_1}(z) + R_{\mu_2, \nu_2}(z).$$

Now, assume that T is a map on the class $\mathcal{M}_c$ of compactly supported probability measures on $\mathbb{R}$, which satisfies the following condition (called *Bożejko property*): if

$$(\mu_1, T\mu_1) \boxplus (\mu_2, T\mu_2) = (\mu, \nu)$$

then $\nu = T\mu$. Defining $\mu_1 \boxplus_T \mu_2 := \mu$ we obtain an associative and commutative operation $\boxplus_T$ on $\mathcal{M}_c$. For example, if $T\mu = \mu$ for every $\mu \in \mathcal{M}_c$ then $\boxplus_T$ is the Voiculescu free convolution $\boxplus$, and if $T\mu = \delta_0$ for every $\mu \in \mathcal{M}_c$ then $\boxplus_T$ becomes the Boolean convolution $\boxplus$.

2 The ϕ -convolution

From now on we fix a compactly supported probability measure ϕ on $\mathbb{R}$ which is infinitely divisible with respect to $\boxplus$. Let

$$G_\phi(z) = \frac{1}{|z - \beta_0|} - \frac{\gamma_0}{|z - \beta_1|} - \frac{\gamma_1}{|z - \beta_2|} - \frac{\gamma_2}{|z - \beta_3|} - \frac{\gamma_3}{|\dots|}$$

be its Cauchy transform. Krystek and Wojakowski [3] defined a convolution $\boxplus_\phi$ in the following way. For $\mu \in \mathcal{M}_c$, we put $T\mu := \phi^{\boxplus V(\mu)}$ (the free power of ϕ). Then T admits Bożejko property (Theorem 7 in Ref. [3]) and we denote $\boxplus_\phi := \boxplus_T$. The authors of Ref.[3] found the related limit measures only in the case when ϕ is either the Wigner or the free Poisson measure.

We are going to exhibit the relation between the Jacobi coefficients of ϕ and those of the limit measures with respect to $\boxplus_\phi$. In particular we prove that a modified version of hypothesis given in Ref. [3] Remark 10 is true.

For $\mu \in \mathcal{M}_c$ and $\lambda > 0$ we define *dilation* of μ by $\mathcal{D}_\lambda \mu(A) := \mu(\lambda^{-1}A)$. Denote by $\gamma_m^{(\lambda)}, \beta_m^{(\lambda)}$ the Jacobi coefficients of the free power $\phi^{\boxplus \lambda}$.

Theorem 2.1 (The central limit theorem) *Assume that $\mu \in \mathcal{M}_c$ satisfies $\int_{\mathbb{R}} t d\mu(t) = 0$ and $\int_{\mathbb{R}} t^2 d\mu(t) = \lambda$. Then the sequence*

$$\xi_{\lambda, N} := \mathcal{D}_{\frac{1}{\sqrt{N}}} \mu \boxplus_\phi \cdots \boxplus_\phi \mathcal{D}_{\frac{1}{\sqrt{N}}} \mu,$$

N times, is $*$ -weakly convergent to the measure $\xi_\lambda \in \mathcal{M}_c$ such that

$$G_{\xi_\lambda}(z) = \frac{1}{z} - \frac{\lambda}{z - \beta_0^{(\lambda)}} - \frac{\gamma_0^{(\lambda)}}{z - \beta_1^{(\lambda)}} - \frac{\gamma_1^{(\lambda)}}{z - \beta_2^{(\lambda)}} - \frac{\gamma_2^{(\lambda)}}{\dots} \quad (5)$$

In particular, ξ is symmetric if and only if ϕ is symmetric.

Proof. By (slightly generalized version of) Ref. [3], Theorem 8 and (4) we have $\frac{1}{G_{\xi_\lambda}(z)} = z - \lambda G_{\phi^{\boxplus \lambda}}(z)$, which proves (5). It remains to use Proposition 1(i). $\square$

Theorem 2.2 (*The Poisson limit theorem*) For $\lambda > 0$ the sequence

$$\rho_{\lambda, N} := \left(\left(1 - \frac{\lambda}{N} \right) \delta_0 + \frac{\lambda}{N} \delta_1 \right) \boxplus_\phi \cdots \boxplus_\phi \left(\left(1 - \frac{\lambda}{N} \right) \delta_0 + \frac{\lambda}{N} \delta_1 \right),$$

N times, is $*$ -weakly convergent to the measure ρ_λ , which satisfies

$$G_{\rho_\lambda}(z) = \frac{1}{z - \lambda} - \frac{\lambda}{z - \beta_0^{(\lambda)} - 1} - \frac{\gamma_0^{(\lambda)}}{z - \beta_1^{(\lambda)}} - \frac{\gamma_1^{(\lambda)}}{z - \beta_2^{(\lambda)}} - \frac{\gamma_2^{(\lambda)}}{\dots} \quad (6)$$

Moreover, the support of ρ_λ is contained in $[0, +\infty)$ if and only if the support of ϕ is contained in $[0, +\infty)$.

Proof. According to Ref. [3], Theorem 9 and formula (4) we have

$$\frac{1}{G_{\rho_\lambda}(z)} = z - \frac{\lambda}{1 - G_{\phi^{\boxplus \lambda}}(z)} = z - \lambda - \frac{\lambda}{\frac{1}{G_{\phi^{\boxplus \lambda}}(z)} - 1},$$

which leads to (6).

Assume that $\text{supp}(\phi) \subseteq [0, +\infty)$, then also $\text{supp}(\phi^{\boxplus \lambda}) \subseteq [0, +\infty)$. Let $\{\lambda_m\}_{m=0}^\infty$ be the sequence of upper Jacobi coefficients of $(\phi^{\boxplus \lambda})_{\text{sym}}$, according to Proposition 1.1(ii). Then the numbers $\lambda'_0 := \lambda$, $\lambda'_1 := 1$ and $\lambda'_k := \lambda_{k-2}$ for $k \geq 2$, are the upper Jacobi coefficients for $(\rho_\lambda)_{\text{sym}}$. On the other hand, if the support of ρ_λ is contained in $[0, \infty)$ and if λ'_m are the upper Jacobi coefficients of $(\rho_\lambda)_{\text{sym}}$ then we have $\lambda'_0 = \lambda$, $\lambda'_1 = 1$ and the numbers λ_{m+2} , $m \geq 0$, are the upper Jacobi coefficients of $(\phi^{\boxplus \lambda})_{\text{sym}}$. $\square$

Remark 2.1 The limit measures ξ_λ and ρ_λ are $\boxplus_\phi$ -infinitely divisible. More generally, let $\mu = \mu(\lambda, u, v)$, with $\lambda > 0$, $u, v \in \mathbb{R}$, denote such a measure μ that

$$G_\mu(z) = \frac{1}{z - u} - \frac{\lambda}{z - \beta_0^{(\lambda)} - v} - \frac{\gamma_0^{(\lambda)}}{z - \beta_1^{(\lambda)}} - \frac{\gamma_1^{(\lambda)}}{z - \beta_2^{(\lambda)}} - \frac{\gamma_2^{(\lambda)}}{\dots}. \quad (7)$$

Then we have $V(\mu) = \lambda$ and

$$\frac{1}{G_\mu(z)} = z - u - \frac{\lambda}{\frac{1}{G_{\phi^{\boxplus \lambda}}(z)} - v} = z - u - \frac{\lambda G_{\phi^{\boxplus \lambda}}(z)}{1 - v G_{\phi^{\boxplus \lambda}}(z)}, \quad (8)$$

which, by 4, leads

$$R_{\mu, \phi^{\boxplus \lambda}}(z) = u + \frac{\lambda z}{1 - v z}. \quad (9)$$

Therefore we have

$$\mu(\lambda_1, u_1, v) \boxplus_\phi \mu(\lambda_2, u_2, v) = \mu(\lambda_1 + \lambda_2, u_1 + u_2, v) \quad (10)$$

and for $t > 0$

$$\mu(\lambda, u, v)^{\boxplus_\phi t} = \mu(t\lambda, tu, v), \quad (11)$$

which means that all the measures $\mu(\lambda, u, v)$ are $\boxplus_\phi$ -infinitely divisible.

Acknowledgment

Research supported by RTN: HPRN-CT-2002-00279, W. Młotkowski is also sponsored by KBN: 1P03A 01330 and by ToK: MTKD-CT-2004-013389. This work was carried out partly during stay of Melanie Hinz in Wrocław (5.03–31.07.2006) and of Wojciech Młotkowski in Greifswald (26.10–5.11.2006).

References

- [1] M. Bożejko, M. Leinert, R. Speicher, *Convolution and limit theorems for conditionally free random variables*, Pacific J. Math. **175** no. 2 (1996), 357–388.
- [2] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach.
- [3] A. D. Krystek, Ł. J. Wojakowski, *Associative convolutions arising from conditionally free convolution*, Infin. Dim. Anal. Quantum Probab. Related Top. **8** (2005), 515–545.