PRODUCT SYSTEMS OF HILBERT SPACES

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We wish to cover the basics of the theory of tensor product systems of Hilbert spaces. Concrete tensor product systems of Hilbert spaces have appeared in various fields of Mathematics and Mathematical Physics much before a formal theory was proposed by W. Arveson. He encountered them [Ar1] in his effort to classify unital semigroups of endomorphisms (E_0-semigroups) of the algebra of all bounded operators on a Hilbert space.

W. Arveson described a way of associating the tensor product systems E_0-semigroups. Other methods were soon discovered (see [Bh1]). In the reverse direction, getting an E_0-semigroup for every product system turned out to be much more complicated[Ar2]. Simpler proofs with more direct approaches are now available ([Sk1-3],[Ar3]).

W. Arveson showed that E_0-semigroups are classified up to cocycle conjugacy by product systems. Then we have the question as to 'How many product systems are there?' and we are lead to classification theory of product systems. This classification scheme depends upon units of the product system. Here units are certain families vectors of the product system which factorize in a simple way. Depending upon availability of units and their richness product systems are put into three classes called type I, II and III. A numerical invariant called the index can also be associated with product systems leading to finer classification. Type I examples come from Fock spaces and they are the paradigm examples. Most of the classification theory and further work is guided by the features of Fock spaces. So a special emphasis will be given to Type I product systems and associated E_0-semigroups. Currently there are several methods of constructing exotic (non type I) product systems are available and we may see them in lectures of V. Liebscher and B. Tsirelson. Independent of the theory of E_0-semigroups and their classification, a dilation theory of Markov Semigroups was being developed by quantum probabilists in order to develop a theory of non-commutative Markov processes[Pa]. These two fields got intimately connected when it was shown [Bh1] that the time shift on weak Markov flows of Markov semigroups yield E_0-semigroups in a very canonical fashion. The theorem then is that Markov semigroups admit unique minimal dilations to E_0 semigroups. This leads to a much better understanding of both Markov semigroups and E_0-semigroups. The notion of product systems comes in handy here as they can be used in the very construction of the dilation. R.T. Powers has constructed exotic product systems using dilation theory[Po]. Extending the theory of 'E_0-semigroups, product systems, dilation of Markov semigroups' to more general C^* or von Neumann algebras would involve in bringing in Hilbert C^*-modules [BS] and we should see them in the lectures of U. Franz and M. Skeide.

Tentatively these lectures could be divided as follows:

- Lecture I: Introduction to endomorphisms, E_0-semigroups and product systems.
- Lecture II: Units, index and type classification of product systems.

- Lecture IV: Dilation theory of Markov Semigroups -II.
- Lecture V: E_0-semigroups from product systems.

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