# INTRODUCTION TO HILBERT C* MODULES

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Hilbert C*-modules are mathematical objects which generalise the notion of a Hilbert space, in that they endow a linear space with an "inner product" which takes values in a C*-algebra. Hilbert C*-modules were first introduced in the work of Kaplansky [Kap53] in 1953, who developed the theory for commutative, unital algebras. In the 1970s the theory was extended to non-commutative C**-algebras independently by Paschke [Pas73] and Rieffel [Rie72, Rie74]. In recent years many interesting applications of Hilbert C*-module theory have been found. Hilbert C*-modules are useful tools in the theory of operator algebras, operator K-theory, group representation theory, and theory of operator spaces, cf. [Lan95, WO93, Ske01]. They also play a central role in Morita equivalence of C*-algebras, K-theory of C*-algebras, and the theory of C*-algebraic quantum groups. Michael Frank's "Bibliography of Hilbert C*-Modules Home Page" (`http://www.imn.htwk-leipzig.de/~mfrank/hilmod.html`) lists about 1456 references.

In Quantum Dynamics, product systems of Hilbert C*-modules where introduced by Bhat and Skeide[BS00], as a generalization of products systems of Hilbert spaces [Arv89]. They are necessary to extend Arveson's theory from $B(H)$ to general C*-algebras. The theory of these products systems and their applications to Quantum Dynamics will be presented in Michael Skeide's lecture in the second week of this School.

The present lecture aims to give a self-contained introduction to Hilbert C*-modules and to provide the prerequisites necessary for Michael Skeide's lecture. Furthermore we plan to discuss applications of Hilbert C*-modules in quantum probability (cf., e.g., [Ske00, Ske01, HKK04, SG07]).

Plan

1. **Lecture:** Introduction to C*-algebras. Definition of Hilbert C*-modules, basic examples, algebra of adjointable operators.
2. **Lecture:** Positive definite kernels, Stinespring factorisation theorem, Kolmogorov decomposition.
3. **Lecture:** Important constructions: tensor product of Hilbert spaces and modules. Morita equivalence.
4. **Lecture:** Further examples and applications (e.g. quantum stochastic calculus)

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