Product systems of Hilbert modules and their applications in quantum dynamics

Michael Skeide

Markov semigroups (semigroups of unital completely positive maps), E_0–semigroups (semigroups of unital endomorphisms), and dilations of the former to the latter, are intimately related to and connected by product systems. This has been illustrated in the lecture of Rajarama Bhat for the case when the semigroups act on $\mathcal{B}(H)$, the algebra of all bounded linear operators on a Hilbert space H.

When the Markov semigroups act on a unital C^*–algebra $\mathcal{B}$, the dilating E_0–semigroups will act on the algebra $\mathcal{B}^a(E)$ of all adjointable operators on a Hilbert $\mathcal{B}$–module E, and the related product systems consist of correspondences over $\mathcal{B}$.

We immediately see a first difference that illustrates how much more flexible this setting is, when we look at the concept of units. Like in the Hilbert space case, units help classifying product systems. But unlike units in Arveson systems (that is, product systems of Hilbert spaces), units in product systems are related to CP-semigroups (semigroups of completely positive maps) or, more generally, to CPD-semigroups (semigroups of completely positive definite kernels). A single unit can generate an interesting product system.

The units that behave most similar to (normalised) units in Arveson systems are central (unital) units (that is, units whose elements commute with the elements of the algebra $\mathcal{B}$). A product system that admits a central unital unit is called spatial. Spatiality of a product systems corresponds to spatiality of the E_0-semigroup from which it is derived. On the level of CP-semigroups the same statement is true only for von Neumann algebras $\mathcal{B}$ (and von Neumann modules). For unital C^*–algebras equivalence of spatiality of a CP-semigroup to that the product system of that CP-semigroup embeds into a spatial one.

Spatiality of both the dynamics and the associated product system of the dynamics is a key property for this school! In fact, when a product system is spatial it is easy to construct a so-called noise having that product system. Noises come along with monotone independent filtrations. (In the $\mathcal{B}(H)$–case also tensor independent.) A monotone noise may always be "blown up" to a free noise. If the product system of a noise is also the product system of another E_0–semigroup, then that E_0–semigroup and the noise are stably cocycle conjugate (that is, suitable amplifications of them are cocycle conjugate). As a consequence, a Markov semigroup allows a Hudson-Parthasarathy dilation (that is, more or less a unital dilation which is a cocycle perturbation of a noise) if and only if it is spatial. This subsummarises all existence results obtained by quantum stochastic calculus. However, this abstract construction does not address the problem if the perturbing cocycle is adapted to a filtration or if it fulfils a quantum stochastic differential equation. But, it tells us that we need not look for a Hudson-Parthasarathy dilation if the Markov semigroup is not spatial.

We give a preliminary outline of our five lectures.

I. Product systems and E_0–semigroups: Although in this school we are mainly interested in product systems of E_0–semigroups that dilate some Markov semigroup, specifically in spatial product systems, for didactic reasons it is convenient to look first at the general relation between product systems and arbitrary E_0-semigroups. In this section we discuss the one-to-one correspondence between product systems (up to isomorphism) and E_0–semigroups (up to stable cocycle conjugacy).

II. Product systems and CP(D)-semigroups: Every CP-semigroup has a product system. If the semigroup is even Markov, then we easily construct a dilation with help of that product system. We do a similar thing also for CPD-semigroups.

III. Spatial product systems: In this lecture we present the basic classification of product systems into spatial and nonspatial, and into types I, II, III. We show that spatial type I systems are Fock, and that type I systems of von Neumann correspondences are spatial automatically. We define an index for spatial product systems and show that this index is additive under a suitable product of spatial product systems.

IV. Spatial dynamics and noises: In this lecture we discuss spatial dilations of spatial Markov semigroups by cocycle perturbations of noises. We briefly sketch the relation with free product systems and free noises.

V. Commutants of von Neumann correspondences and representations of product systems: Actually, Arveson's original approach to Arveson systems from E_0–semigroups on $\mathcal{B}(H)$ is dual to Bhat's approach, which is the basis for our discussion of the general case. (In fact, the two Arveson systems are anti-isomorphic and need not be isomorphic.) In the general context, the duality is described as the commutant on the level of product systems of von Neumann correspondences.