Commutative Relations between Boolean and Free Convolutions

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Outline

Basic Definitions

Non-Commutative Probability Spaces

A non-commutative probability is a pair $(\mathscr{A},\varphi)$, where $\mathscr{A}$ is a unital algebra and $\varphi : \mathscr{A} \to \mathbb{C}$ is a linear functional s.t. $\varphi(1) = 1$. When $\mathscr{A}$ is a C^*-algebra and φ is positive we call $\mathscr{A}$ a C^*-probability space. In this frame we will talk about:

(non-commutative) random variables: $a \in \mathscr{A}$

normal elements: $a \in \mathscr{A}$ s. t. $a^*a = aa^*$

self-adjoint: $a \in \mathscr{A}$ s. t. $a = a^*$

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*-distributions

We will be interested in the **moments** of an element $a \in \mathscr{A}$:

$$\varphi(a^{m_1}(a^*)^{n_1}\dots(a^*)^{n_k})$$

If $a \in \mathscr{A}$ is normal and μ_a is a probability measure on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} z^l \bar{z}^k \mathrm{d}\mu(z) = \varphi\left(a^l (a^*)^k\right),$$

we call μ_a the $*$-**distribution** of a.
If $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$ is normal then the $*$-distribution of a exists and is unique.

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Notation

- We will denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$.
- We will denote by $\mathbb{R}^+$ the positive real line $[0,\infty)$.
- $\mathscr{P}(\mathbb{H})$ are the probability measures on $\mathbb{H}$. $(\mathbb{H} = \mathbb{C}, \mathbb{T}, \mathbb{R}, \mathbb{R}^+)$
- $\mathscr{P}(\mathbb{R})$ and $\mathscr{P}(\mathbb{R}^+)$ will be also called $\mathscr{M}$ and $\mathscr{M}^+$.
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Free Independence

The unital sub-algebras $(\mathscr{A}_i)_{i\in I}$ of $\mathscr{A}$ are called **freely independent** (free) if for any $k \in \mathbb{N}$, we have

$$\varphi(a_1 \dots a_k) = 0,$$

whenever:

i) $a_i \in \mathscr{A}_{j(i)}, j(1) \neq j(2) \neq \cdots \neq j(k)$ and

ii) $\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_k) = 0.$

Random variables $(a_i)_{i\in I}$ are **free** if the unital algebras generated by them are free.

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Additive convolutions

Free Additive Convolution

Definition

Let $a, b \in \mathscr{A}^s$ be free random variables and denote by μ and ν the $*$-distributions of a and b respectively. The **free additive convolution** of μ and ν is the $*$-distribution of $a+b$ and is denoted by $\mu \boxplus \nu$.

- The operation $\mu \boxplus \nu$ is associative and commutative.
- For $t \geq 1$, $\mu^{\boxplus t}$ is a well defined probability measure.

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Free infinite divisibility

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Let $\mu \in M$. We say that μ is **freely infinitely divisible** if for every $n \in \mathbb{N}$, there exisits μ_n s. t.

$$\mu = \mu_n \boxplus \cdots \boxplus \mu_n$$

- We denote by $ID(\boxplus)$ the class of free infinitely divisible measures. If $\mu \in ID(\boxplus)$ we also say $\boxplus$-infinitely divisible.
- $\mu \in ID(\boxplus)$ is equivalent to the existence of a semigroup $\mu_s \boxplus \mu_t = \mu_{t+s}$ with $\mu_1 = \mu$.

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First Commutation Relation

Theorem (Bellinschi, Nica)

Let μ be a probability measure on $\mathbb{R}$. Then for $p \geq 1$ and $q > (p-1)/p$ we have

$$(\mu^{\boxplus p})^{\uplus q} = (\mu^{\uplus q'})^{\boxplus p'}$$

where $q' = 1 - p + pq$ and $p'q' = pq$.

Note that $q' > 0$ and $p' \geq 1$ so the powers in the RHS indeed make sense.

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Another Semigroup

Theorem (Bellinschi, Nica.)

For every $t \geq 0$ one can define a one to one map $\mathbb{B}_t : \mathscr{M} \to \mathscr{M}$ by

$$\mathbb{B}_t(\mu) = (\mu^{\boxplus 1+t})^{\uplus 1/(1+t)}$$

Moreover, the maps $\{\mathbb{B}_t \mid t \geq 0\}$ form a semigroup:

$$\mathbb{B}_{t+s} = \mathbb{B}_t \circ \mathbb{B}_s = \mathbb{B}_s \circ \mathbb{B}_t$$

Proof.

Injectvity follows since the maps $\mu \mapsto \mu^{\boxplus(1+t)}$ and $\mu \mapsto \mu^{\uplus(1/(1+t)}$ are themselves one to one. □

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Additive convolution

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Proof.

Use directly last propostion

$$\begin{aligned}
\mathbb{B}_t(\mathbb{B}_s(\mu)) &= \mathbb{B}_t((\mu^{\boxplus 1+s})^{\uplus 1/(1+s)}) \\
&= ([(\mu^{\boxplus 1+s})^{\uplus 1/(1+s)}]^{\boxplus 1+t})^{\uplus(1/(1+t))} \\
&= ([(\mu^{\boxplus 1+s})^{\boxplus (t+s+1)/(1+s)}]^{\uplus(1+t)/(t+s+1)})^{\uplus(1/(1+t))} \\
&= ([(\mu^{\boxplus (t+s+1)/(1+s)})^{\uplus(1/(1+t+s))} \\
&= \mathbb{B}_{t+s}(\mu)
\end{aligned}$$

□

Additive convolutions

Properties of $\mathbb{B}_t$

- $\mathbb{B}_1$ coincides with the Boolean-to-free Bercovici Pata bijection.
- $\mathbb{B}_s(\mu)$ is $\boxplus-$infinitely divisible for all $s \geq 1$.
- $\mathbb{B}_s(\mu \boxtimes \nu) = \mathbb{B}_s(\mu) \boxtimes \mathbb{B}_s(\nu)$, for all $\mu \in \mathcal{M}$ and $\nu \in \mathcal{M}^+$.
- $\mathbb{B}_s$ sends measures in $\mathcal{M}^+$ to measures in $\mathcal{M}^+$.
- $\mathbb{B}_s(\mu)$ has at most $[1/s] + 1$ atoms.

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Divisibility Indicator

Definition

For $\mu \in \mathscr{M}$ we denote

$$\varphi(\mu) := sup\{t \geq 0 \mid \mu \in \mathbb{B}_t(\mathscr{M})\}$$

We will call $\varphi(\mu)$ the $\boxplus$-divisibility indicator of μ.

Additive convolutions

Properties inf div indicator

- $\varphi(\mu) \geq 1$ if and only if μ is freely infinitely divisible.
- If $0 < \varphi(\mu) < 1$ then for $t > 1 - \varphi(\mu)$, the probability measure $\mu^{\boxplus t}$ exists.
- If $\varphi(\mu) = p$ then $\exists \nu$ such that $\mathbb{B}_s(\nu) = \mu$.
- $\varphi(\mathbb{B}_s(\mu)) = s + \varphi((\mu))$.
- $\varphi(\mu)$ is invariant under shifts and dilations. ($\mu_t(A) = \mu(A+t)$ and $D_t(\mu)(A) = \mu(tA)$).

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Properties of the divisibility indicator.

Theorem (Hasebe, A.)

Let $\mu \in \mathscr{M}$. Then, for $s > 0$, we have

$$\varphi(\mu^{\uplus s}) = \frac{1}{s}\varphi(\mu).$$

Proof.

Suppose $\varphi(\mu) = t$. Then $\mu = \mathbb{B}_t(\nu)$ and we have

$$\mu^{\uplus s} = \left(\left(\nu^{\boxplus(1+t)}\right)^{\uplus \frac{s+t}{1+t}}\right)^{\uplus \frac{s}{s+t}} = \left(\left(\nu^{\uplus s}\right)^{\boxplus \frac{s+t}{s}}\right)^{\uplus \frac{s}{s+t}}$$

$$= \left(\left(\nu^{\uplus s}\right)^{\uplus 1+t/s}\right)^{\uplus \frac{1}{1+t/s}} = \mathbb{B}_{t/s}(\nu^{\uplus s})$$

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Properties of the divisibility indicator.

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Let $\mu \in \mathcal{M}$. Then, for $s > 0$, we have

$$\varphi(\mu^{\uplus s}) = \frac{1}{s}\varphi(\mu).$$

Proof.

Suppose $\varphi(\mu) = t$. Then $\mu = \mathbb{B}_t(\nu)$ and we have

$$\mu^{\uplus s} = \left((\nu^{\boxplus(1+t)})^{\uplus \frac{s+t}{1+t}}\right)^{\uplus \frac{s}{s+t}} = \left((\nu^{\uplus s})^{\boxplus \frac{s+t}{s}}\right)^{\uplus \frac{s}{s+t}}$$

$$= \left((\nu^{\uplus s})^{\uplus 1+t/s}\right)^{\uplus \frac{1}{1+t/s}} = \mathbb{B}_{t/s}(\nu^{\uplus s})$$

Additive convolutions

Divisibility Indicator

Corollary (Bozejko's Conjecture)

If μ *is* $\boxplus-$*infinitely divisible, then so is* $\mu^{\uplus t}$ *for* $0 < t \leq 1$.

Corollary

If μ *is a Boolean stable law, then either* $\varphi(\mu) = 0$ *or* $\varphi(\mu) = \infty$.

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If μ *is a free stable law, then either* $\varphi(\mu) = 1$ *or* $\varphi(\mu) = \infty$.

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Additive convolutions

Divisibility Indicator

Definition

For $\mu \in \mathscr{M}$ we denote

$$\varphi(\mu) := sup\{t \geq 0 \mid \mu \in \mathbb{B}_t(\mathscr{M})\}$$

Corollary

For $\mu \in \mathscr{M}$ we have

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Multiplicative Convolutions on $\mathbb{R}$

Free Multiplicative Convolution

Definition

Let $a, b \in \mathscr{A}$ be positive free random variables and denote by μ and ν the $*$-distributions of a and b respectively. The **free multiplicative convolution** of μ and ν is the distribution $a^{1/2}ba^{1/2}$ and is denoted by $\mu \boxtimes \nu$.

- The moments of $a^{1/2}ba^{1/2}$ and are ab the same.
- The operation $\boxtimes$ is associative and commutative
- For $t \geq 1$, $\mu^{\boxtimes t}$ is a well defined probability measure.

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Multiplicative Convolutions on $\mathbb{R}$

Infinite divisibility

Definition

Let $\mu \in M$. We say that μ is $\boxtimes$-**infinitely divisible** if for every $n \in \mathbb{N}$, there exisits μ_n s. t.

$$\mu = \mu_n \boxtimes \cdots \boxtimes \mu_n$$

We denote by $ID(\boxtimes)$ the class of free infinitely divisible measures.

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Boolean Multiplicative Convolution

Definition

Let $a, b \in \mathscr{A}$ be positive Boolean independent random variables and denote by μ and ν the $*$-distributions of $a+1$ and $b+1$ respectively. The **Boolean multiplicative convolution** of μ and ν is the distribution $(a+1)(b+1)$ and is denoted by $\mu \uplus \nu$

- The operation $\mu \uplus \nu$ is associative and commutative.
- For $t < 1$, $\mu^{\uplus t}$ is a well defined probability measure. (Every measure is $\uplus$-infinitely divisible)
- For every μ there exist some $N > 0$ such that $\mu^{\uplus N}$ is NOT a probability measure!!!

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Multiplicative Convolutions on $\mathbb{R}$

Second Commutation Relation

Theorem (Hasebe, A.)

Let μ be a probability measure on $\mathbb{R}$. Then for $p \geq 1$ and $1 \geq q > (p-1)/p$ we have

$$(\mu^{\boxtimes p})^{\uplus q} = (\mu^{\uplus q'})^{\boxtimes p'}$$

where $q' = 1 - p + pq$ and $p'q' = pq$.

- This is exactly the same as in the additive case except for the restriction that $q \leq 1$.

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Multiplicative Convolutions on $\mathbb{R}$

Semigroup

Theorem (Hasebe, A.)

For every $t \geq 0$ one can define a map $\mathbb{M}_t : M \to \mathscr{M}$ by

$$\mathbb{M}_t(\mu) = (\mu^{\boxtimes 1+t})^{\uplus 1/(1+t)}$$

Moreover, the maps $\{\mathbb{B}_t \mid t \geq 0\}$ form a semigroup:

$$\mathbb{M}_{t+s} = \mathbb{M}_t \circ \mathbb{M}_s = \mathbb{M}_s \circ \mathbb{M}_t$$

Multiplicative Convolutions on $\mathbb{R}$

Divisibility Indicator

Definition

For $\mu \in M$ we denote

$$\theta(\mu) := sup\{t \geq 0 \mid \mu \in \mathbb{M}_t(\mathscr{M})\}$$

We will call $\theta(\mu)$ the $\boxtimes$-divisibility indicator of μ.

- μ is freely infinitely divisible if $\theta(\mu) \geq 1$.
- $\theta(\mathbb{B}_s(\mu)) = s + \theta((\mu))$.

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Multiplicative Convolutions on $\mathbb{T}$

Commutation Relation

Theorem (Hasebe, A.)

Let $\mu \in ID(\uplus, \mathbb{T})$. Then, for $p \geq 1$ and $q > (p-1)/p$, we have

$$(\mu^{\boxtimes p})^{\uplus q} = (\mu^{\uplus q'})^{\boxtimes p'}$$

where $q' = 1 - p + pq$ and $p'q' = pq$.

Multiplicative Convolutions on $\mathbb{T}$

Commutation Relation

Theorem (Hasebe, A.)

Let $\mu \in ID(\mathbb{U},\mathbb{T})_0$. Then for $p \geq 1$ and $q > (p-1)/p$ we have

$$(\mu^{\boxtimes[\arg m_1(\mu)]^p})^{\mathbb{U}[p \arg m_1(\mu)]^q} = (\mu^{\mathbb{U}[\arg m_1(\mu)]^{q'}})^{\boxtimes[q' \arg m_1(\mu)]^{p'}}$$

where $q' = 1 - p + pq$ and $p'q' = pq$. Moreover

1)$(\mu^{\boxtimes[\arg m_1(\mu)]^t})^{\boxtimes[t \arg m_1(\mu)]^s} = \mu^{\boxtimes[\arg m_1(\mu)]^{st}}$

2)$(\mu^{\mathbb{U}[\arg m_1(\mu)]^t})^{\mathbb{U}[t \arg m_1(\mu)]^s} = \mu^{\mathbb{U}[\arg m_1(\mu)]^{st}}$

Multiplicative Convolutions on $\mathbb{T}$

First Commutation Relation

Definition (Hasebe, A.)

For every $t \geq 0$ one can define a map $\mathbb{K}_t : ID(\uplus, \mathbb{T}) \mapsto ID(\uplus, \mathbb{T})$ by

$$\mathbb{K}_t(\mu) = (\mu^{\boxtimes 1+t})^{\uplus 1/(1+t)}$$

Moreover, the maps $\{\mathbb{K}_t \mid t \geq 0\}$ form a semigroup:

$$\mathbb{K}_{t+s} = \mathbb{K}_t \circ \mathbb{K}_s = \mathbb{K}_s \circ \mathbb{K}_t$$

Multiplicative Convolutions on $\mathbb{T}$

Divisibility Indicator

Definition

For $\mu \in M$ we denote

$$\theta(\mu) := sup\{t \geq 0 \mid \mu \in \mathbb{K}_t(ID(\mathbb{U}, \mathbb{T})\}$$

We will call $\theta(\mu)$ the $\boxtimes$-divisibility indicator of μ.

Multiplicative Convolutions on $\mathbb{T}$

Divisibility Indicator

- μ is freely infinitely divisible if and only if $\theta(\mu) \geq 1$.
- $\theta(\mu^{\uplus s})) = \frac{1}{s}\theta((\mu))$.
- If $0 < \theta(\mu) < 1$ then for $t > 1 - \varphi(\mu)$, the probability measure $\mu^{\boxtimes t}$ exists.
- $\theta(\mu^{\boxtimes s}) - 1 = \frac{1}{s}(\theta(\mu) - 1)$.
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Multiplicative Convolutions on $\mathbb{T}$

Divisibility Indicator

Corollary

For $\mu \in (ID(\mathbb{U}, \mathbb{T})$ *we have*

$$\theta(\mu) = sup\{t \geq 0 \mid \mu^{\mathbb{U}t} \text{ is } \boxtimes - \text{infinitely divisible}\}$$

Corollary

If μ *is* $\boxtimes$*-infinitely divisible, then so is* $\mu^{\mathbb{U}t}$ *for* $0 < t \leq 1$.

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Further Relations

Commuting relations.

Theorem

The following commutation relations holds for $\mu \in M+$.

- $(\mu^{\boxplus t})^{\boxtimes s} = D_{t^{s-1}}(\mu^{\boxtimes s})^{\boxplus t}$ for $t > 1$ and s such that $\mu^{\boxtimes s}$ exists.
- $(\mu^{\uplus t})^{\boxtimes s} = D_{t^{s-1}}(\mu^{\boxtimes s})^{\uplus t}$ for $t > 0$ and s such that $\mu^{\boxtimes s}$ exists.
- $(\mu^{\uplus t})^{s} = (\mu^{\Cup s})^{\uplus t^{s}}$ for $t > 0$ and s such that $\mu^{\Cup s}$ exists.

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Further Relations

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If $\mu \in \mathscr{P}(\mathbb{R}^{+})$ is $\boxtimes$-infinitely divisible, then so is $\mu^{\boxplus t}$ for $t > 1$ and $\mu^{\uplus s}$ for .$s > 0$

Remark. It is NOT true that if μ is $\boxplus$-infinitely divisible then $\mu^{\boxtimes s}$.

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Thanks..