Additive Deformations and Lévy Processes on Hopf Algebras

Malte Gerhold

Institut für Mathematik und Informatik, Universität Greifswald

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joint work with Kietzmann, Lachs

Content

Heisenberg Algebra

Def. (Heisenberg algebra)

algebra $\mathcal{A}$ with generators a, a^* and commutation relation

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$\mathcal{A}$ bialgebra?

no bialgebra structure s.t.

$$\Delta a^{(*)} = a^{(*)} \otimes \mathbb{1} + \mathbb{1} \otimes a^{(*)},$$

since

$$[\Delta a, \Delta a^*] = [a, a^*] \otimes \mathbb{1} + \mathbb{1} \otimes [a, a^*] = 2\mathbb{1} \otimes \mathbb{1} \neq \Delta [a, a^*]$$

A Way Out

Idea

define comultiplication as algebra homomorphism

$$\Delta : \mathcal{A}_{s+t} \to \mathcal{A}_s \otimes \mathcal{A}_t ,$$

- $\mathcal{A}_t$ is algebra with generators a, a^* and commutation relation

$$[a, a^*]_t = t\mathbb{1}$$

- relations respected:

$$[a, a^*]_s \otimes \mathbb{1} + \mathbb{1} \otimes [a, a^*]_t = s\mathbb{1} \otimes \mathbb{1} + t\mathbb{1} \otimes \mathbb{1} = \Delta([a, a^*]_{t+s})$$

The Fermi Case

The algebra

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We have to change the multiplication in the tensor product!

$$\Delta(a)\Delta(a^*) + \Delta(a^*)\Delta(a) \stackrel{!}{=} (aa^* + a^*a) \otimes \mathbb{1} + \mathbb{1} \otimes (aa^* + a^*a)$$

Braidings

Def. (braiding)

Invertible linear map $\beta : V \otimes V \to V \otimes V$, s.t.

$$(\beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta) \circ (\beta \otimes \mathrm{id}) = (\mathrm{id} \otimes \beta) \circ (\beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \beta)$$

(Young Baxter equation)

Def. (β-invariance)

$R : V^{\otimes m} \to V^{\otimes n}$

$$(R \otimes \mathrm{id}) \circ \beta_{1,m} = \beta_{1,n} \circ (\mathrm{id} \otimes R)$$

β-compatible: β- and β^{-1}-invariant

Braided Structures

Def. (algebra, coalgebra, bialgebra)

Braided algebra: μ and $\mathbb{1}$ are β-compatible
braided coalgebra: Δ and δ are β-compatible
braided bialgebra: braided algebra, braided coalgebra, Δ and δ are algebra homomorphisms.

Theorem

Let $\mathcal{B}$ be a braided algebra/coalgebra/bialgebra. Then $\mathcal{A} \otimes \mathcal{A}$ is an algebra/coalgebra/bialgebra with

- braiding $\beta_{2,2} = (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\beta \otimes \beta) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id})$
- multiplication $M = (\mu \otimes \mu) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id})$
- unit $\mathbb{1} \otimes \mathbb{1}$
- comultiplication $\Lambda = (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$
- counit $\delta \otimes \delta$

What About $*$-Structures?

Def. (braided $*$-bialgebra)

braided bialgebra with involution $*$, s.t.

$$(\beta \circ (* \otimes *) \circ \tau)^2 = \mathrm{id} \otimes \mathrm{id}$$

Remark

due to Franz-Schott-Schürmann

$$(a \otimes b)^* = ((a \otimes \mathbb{1})(\mathbb{1} \otimes b))^* = ((\mathbb{1} \otimes b^*)(a^* \otimes \mathbb{1})) = \beta(b^* \otimes a^*)$$

$\rightarrow *_{\otimes} = \beta \circ (* \otimes *) \circ \tau$ only involution on the tensor product that behaves well with inclusion and multiplication

Convolution

Def. (convolution)

$\mathcal{C}$ coalgebra, $\mathcal{A}$ algebra

- for $R, M : \mathcal{C} \to \mathcal{A}$ define $R \star M := \mu_{\mathcal{A}} \circ (R \otimes M) \circ \Delta_{\mathcal{C}}$
- for $\varphi, \psi : \mathcal{B} \to \mathbb{C}$ this simplifies to $\varphi \star \psi := (\varphi \otimes \psi) \circ \Delta$

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Def. (pointwise continuous convolution semigroup)

- $(\varphi_t)_{t \in \mathbb{R}}$
- $\varphi_t(a) \xrightarrow{t \to 0} \delta(a) \quad \forall a \in \mathcal{B}$
- $\varphi_t \star \varphi_s = \varphi_{t+s} \quad \forall t, s$

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The following holds

- $\psi := \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t \Big|_{t=0}$ exists pointwise
- $\varphi_t = \exp_\star(t\psi) = \delta + t\psi + \frac{t^2}{2} \psi \star \psi + \dots$

as a consequence of the fundamental theorem for coalgebras

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A Question

Is this a subproduct system of $*$-algebras?

The Generator of an Additive Deformation

Thm. (J. Wirth, 2002; G.-Kietzmann-Lachs)

1-1-correspondence between additive deformations and generators via the equations

$$L = \frac{\mathrm{d}}{\mathrm{d}t}\delta \circ \mu_t \Big|_{t=0} \qquad \mu_t = \mu \star \exp_\star(tL)$$

Def. (generator of an additive deformation)

- $L(\mathbb{1} \otimes \mathbb{1}) = 0$ (L is normalized)
- $L \star \mu = \mu \star L$ (L is commuting)
- $\delta \otimes L + L \circ (\mathrm{id} \otimes \mu) = L \otimes \delta + L \circ (\mu \otimes \mathrm{id})$ (L is 2-cocycle)
- $L \circ (* \otimes *) = \overline{L} \circ \tau$ (L is hermitian)

Proof

- recall the condition

$$\Delta \circ \mu_{t+s} = \underbrace{(\mu_t \otimes \mu_s) \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id})}_{\text{multiplication on } \mathcal{B}_t \otimes \mathcal{B}_s} \circ (\Delta \otimes \Delta)$$

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- applying the counit $\delta \otimes \delta$ to the last condition yields

$$\delta \circ \mu_{t+s} = ((\delta \circ \mu_t) \otimes (\delta \circ \mu_s)) \circ \underbrace{(\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)}_{\text{comultiplication on } \mathcal{B} \otimes \mathcal{B}}$$

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- $(\delta \circ \mu_t)_{t \geq 0}$ pointwise continuous convolution semigroup $\rightarrow \delta \circ \mu_t = \exp_\star(tL)$

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- properties of L follow by differentiation (exercise)
- other direction

Two Example Calculations

a, b grouplike, i.e. $\Delta(a) = a \otimes a$, $\Delta(b) = b \otimes b$

if $\beta(a \otimes b) = b \otimes a$ then $\Lambda(a \otimes b) = a \otimes b \otimes a \otimes b$ and

$$\mu_t(a \otimes b) = \mu_0(a \otimes b)e^{tL(a \otimes b)}$$

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a, b primitive, i.e. $\Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a$

$\Lambda(a \otimes b) = a \otimes b \otimes \mathbb{1} \otimes \mathbb{1} + a \otimes \mathbb{1} \otimes \mathbb{1} \otimes b + \mathbb{1} \otimes b \otimes a \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes a \otimes b$ and

$$\mu_t(a \otimes b) = \mu_0(a \otimes b) + tL(a \otimes b)\mathbb{1}$$

Thm. (J. Wirth, 2002;Franz-Schott-Schürmann; G.-Kietzmann-Lachs)

Given:

- braided $*$-bialgebra $\mathcal{B}$
- additive deformation $(\mu_t)_{t\in\mathbb{R}}$
- $\psi : \mathcal{B} \to \mathbb{C}$ be a hermitian, β-invariant linear functional with $\psi(\mathbb{1}) = 0$

TFAE

(*i*) $\varphi_t := \mathrm{e}_\star^{t\psi}$ is a state on $\mathcal{B}_t$ for all $t \geq 0$, i.e. $\varphi_t(\mathbb{1}) = 1$ and $\varphi_t \circ \mu_t\,(b^* \otimes b) \geq 0$ for all $b \in \mathcal{B}$,

(*ii*) ψ is L-conditionally positive, i.e. $(\psi \circ \mu + L)(b^* \otimes b) \geq 0$ for all $b \in \ker \delta$

Proof of the Easy Direction

Assume $\varphi_t := \mathrm{e}_\star^{t\psi}$ is a state on $\mathcal{B}_t$

- $t \mapsto \varphi_t \circ \mu_t\,(c^* \otimes c)$ is positive for $t \geq 0$

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- for $c \in \ker \delta$

$$\varphi_0 \circ \mu_0\,(c^* \otimes c) = (\delta \otimes \delta)(c^* \otimes c) = |\delta(c)|^2 = 0.$$

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- for $c \in \ker \delta$

$$\varphi_0 \circ \mu_0\,(c^* \otimes c) = (\delta \otimes \delta)(c^* \otimes c) = |\delta(c)|^2 = 0.$$

- so $\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t \circ \mu_t(b^* \otimes b)\big)\Big|_{t=0} = (\psi \circ \mu + L)(b^* \otimes b) \geq 0$

Preparations for the Converse Proof

Lemma

Given

- $M, K : \mathcal{B} \to \mathbb{C}$
- M is β-invariant or K is β^{-1}-invariant

then $\widetilde{M \star K} = \widetilde{M} \circledast \widetilde{K}$, where $\circledast = (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$ is usual convolution and $\widetilde{M} := M \circ (* \otimes \mathrm{id})$

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Thm. (Schürmann)

Given

- coalgebra $\mathcal{C}$
- hermitian sesquilinear form $\widetilde{K} : \overline{\mathcal{C}} \otimes \mathcal{C} \to \mathbb{C}$

TFAE

- $\mathrm{e}_{\circledast}^{t\widetilde{K}}(\overline{c} \otimes c) \geq 0$ for all $c \in \mathcal{C}, t \geq 0$,
- $\widetilde{K}(\overline{c} \otimes c) \geq 0$ for all $c \in \ker \delta$.

Corollary (SC for braided $*$-coalgebras)

Given

- β braided $*$-coalgebra $\mathcal{C}$
- β-invariant, hermitian bilinear form K on $\mathcal{C}$

TFAE

- $\mathrm{e}_\star^{tK}(c^* \otimes c) \geq 0$ for all $c \in \mathcal{C}, t \geq 0$,
- $K(c^* \otimes c) \geq 0$ for all $c \in \ker \delta$.

End of Proof

Given

- β-braided $*$-bialgebra $\mathcal{B}$
- additive deformation $(\mu_t)_{t\in\mathbb{R}}$ with generator L
- L-conditionally positive hermitian linear functional ψ

then $K := \psi \circ \mu + L$ is a hermitian, conditionally positive bilinear form on $\mathcal{B}$, so

$$0 \leq \mathrm{e}_\star^{tK}(c^* \otimes c) = \mathrm{e}_\star^{t\psi\circ\mu+tL}(c^* \otimes c) = \mathrm{e}_\star^{t\psi\circ\mu} \star \mathrm{e}_\star^{tL}(c^* \otimes c)$$
$$= \mathrm{e}_\star^{t\psi} \circ (\mu \star \mathrm{e}_\star^{tL})(c^* \otimes c) = \varphi_t \circ \mu_t(c^* \otimes c),$$

since $(\psi \circ \mu) \star L = \psi \circ (L \star \mu) = \psi \circ (\mu \star L) = L \star (\psi \circ \mu)$ and μ is a coalgebra homomorphism.

The SC Yields Processes

$T_t = (\mathrm{id} \star e_\star^{t\psi}) : B_t \to B$ are completely positive
inductive limit construction yields

- $j_{s,t} : B_{t-s} \to \mathcal{A}$ $*$-algebra homomorphisms
- $j_{s,t} \star j_{t,r} = j_{s,r}$
- $\Phi \circ j_{s,t} = e_\star^{(t-s)\psi}$

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In the case $\beta = \tau$ these processes are solutions of quantum stochastic differential equations.In general?

Once more the Heisenberg Algebra

Given:

- polynomial algebra in two commuting indeterminates $\mathbb{C}[x, x^*]$
- bialgebra structure with primitive comultiplication

For an additive deformation $\mu_t = \mu \star \exp_\star(tL)$

$$[x, x^*]_t = \mu \star \exp_\star(tL)(x \otimes x^* - x^* \otimes x) = tL(x \otimes x^* - x^* \otimes x)\mathbb{1}$$

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Proposition

$L(x \otimes x^*) = 1/2$, $L(x^* \otimes x) = -1/2$ and $L(M \otimes N) = 0$ on all other monomials defines a generator of an additive deformation s.t. $\mathcal{B}_t \cong \mathcal{A}_t$. If L' is another generator with $L(x \otimes x^* - x^* \otimes x) = 1$ there is a semigroup of isomorphisms with $\Phi_t : \mathcal{B}'_t \to \mathcal{B}_t$ for all t.

And the Fermi Case

Given:

- polynomial algebra in two commuting **noncommuting** $\mathbb{C}\langle x, x*\rangle$
- bialgebra structure with primitive comultiplication

Introduce braiding: $\beta(M \otimes N) = (-1)^{deg(M)deg(N)} N \otimes M$ and divide by $I = Ideal(xx^* + x^*x)$ For an additive deformation $\mu_t = \mu \star \exp_\star(tL)$

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- β^{-1} – *invariance* of multiplication implies $\beta(x \otimes x) = q^{-1}x \otimes x$

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- $\Delta(xx^* - qx^*x - 1) \in I \otimes B + B \otimes I$ implies $\beta(x^* \otimes x) = q^{-1} x \otimes x^*$
- β-invariance of multiplication implies $\beta(x \otimes x) = qx \otimes x$
- $\beta^{-1} - invariance$ of multiplication implies $\beta(x \otimes x) = q^{-1} x \otimes x$

What about q?

no chance when $\beta(x \otimes x^*) = qx^* \otimes x$

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so $q^2 = 1$!

Further Results

- B has an antipode $\Rightarrow \exists S_t$ with $\mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta = \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta = \delta \mathbb{1}$
- characterization of trivial deformations
- decomposition of Lévy-processes into gaussian and nondeformed part

Questions

- Where else does this structure appear?
- Can q-brownian motion be described in a similar way?
- Do the Lévy-processes arising from deformation have special properties?

Thank You