Generalizing Minkowski's determinantal inequality

(joint work with J.-C. Bourin)

Fumio Hiai (Tohoku University)

2011, March (Greifswald)

Plan

1. **Minkowski's inequalities with a convex or concave function**
2. **Subadditivity/superadditivity inequalities for symmetric (anti-)norms**
3. **Characterization of convexity/concavity of certain trace functions**
4. **Minkowski's inequalities in the semifinite von Neumann algebra setting**

Reference

- **J.-C. Bourin and F. H., Norm and anti-norm inequalities for positive semi-definite matrices, to appear in** *Internat. J. Math.* **(arXiv:1012.5171)**

1. Minkowski's inequalities with a convex or concave function

Notations

- $\mathbb{M}_n$: the $n \times n$ complex matrices
- $\mathbb{M}_n^+$: the $n \times n$ positive semidefinite matrices
- $\mathbb{M}_n\{\Omega\}$: the set of Hermitian matrices $A \in \mathbb{M}_n$ with spectra in an interval Ω

Minkowski's determinantal inequality For $A, B \in \mathbb{M}_n^+$,

$$\det{}^{1/n}(A+B) \geq \det{}^{1/n}A + \det{}^{1/n}B,$$

i.e.,

$$\left\{\prod_{j=1}^{n} \lambda_j(A+B)\right\}^{1/n} \geq \left\{\prod_{j=1}^{n} \lambda_j(A)\right\}^{1/n} + \left\{\prod_{j=1}^{n} \lambda_j(B)\right\}^{1/n},$$

where $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A. ($\det^{1/n}$ is superadditive and concave on $\mathbb{M}_n^+$)

Rotfel'd trace inequality

When $f : [0,\infty) \to [0,\infty)$ **is concave, for** $A, B \in \mathbb{M}_n^+$**,**

$$\mathrm{Tr}\, f(A+B) \le \mathrm{Tr}\, f(A) + \mathrm{Tr}\, f(B)$$

Aujla-Bourin (2007)

If f **is monotone concave on** $[0,\infty)$ **with** $f(0) \ge 0$ **(resp., monotone convex on** $[0,\infty)$ **with** $f(0) \le 0$**), then for** $A, B \in \mathbb{M}_n^+$**,**

$$f(A+B) \le U f(A) U^* + V f(B) V^* \quad \textbf{(resp., } f(A+B) \ge U f(A) U^* + V f(B) V^*\textbf{)}$$

for some unitary $U, V \in \mathbb{M}_n$**.**

Bourin (2005, 2011?)

If f **is concave on an interval** Ω**, then for** $A, B \in \mathbb{M}_n\{\Omega\}$**,**

$$f\left(\frac{A+B}{2}\right) \ge \frac{1}{2}\left\{U \frac{f(A)+f(B)}{2} U^* + V \frac{f(A)+f(B)}{2} V^*\right\}$$

for some unitary $U, V \in \mathbb{M}_n$**.**

Extensions of the Rotfel'd and Minkowski's inequalities

From the above inequalities involving unitary matrices, we have

- **When** $f : [0, \infty) \to [0, \infty)$ **is concave, for** $A, B \in \mathbb{M}_n^+$**,**

  $$\|f(A+B)\| \le \|f(A)\| + \|f(B)\|$$

  for any symmetric (or unitarily invariant) norm $\|\cdot\|$**.**

- **When** $g : [0, \infty) \to [0, \infty)$ **is convex with** $g(0) = 0$**, for** $A, B \in \mathbb{M}_n^+$**,**

  $$\det{}^{1/n} g(A+B) \ge \det{}^{1/n} g(A) + \det{}^{1/n} g(B).$$

- **When** $f : [0, \infty) \to [0, \infty)$ **is concave, for** $A, B \in \mathbb{M}_n^+$**,**

  $$\det{}^{1/n} f\left(\frac{A+B}{2}\right) \ge \frac{\det{}^{1/n} f(A) + \det{}^{1/n} f(B)}{2}.$$

2. Subadditivity/superadditivity inequalities for symmetric (anti-)norms

Definition

A functional $A \in \mathbb{M}_n^+ \mapsto \|A\|_! \in [0, \infty)$ **is called a symmetric anti-norm if**

- **homogeneous and concave (equivalently, superadditive), i.e.,**

$$\|\lambda A\|_! = \lambda \|A\|_!, \quad \|A + B\|_! \geq \|A\|_! + \|B\|_!$$

  for any $\lambda \geq 0$ **and** $A, B \in \mathbb{M}_n^+$**,**

- **unitarily invariant (or symmetric), i.e.,**

$$\|A\|_! = \|UAU^*\|_!$$

  for any unitary $U \in \mathbb{M}_n$ **and** $A \in \mathbb{M}_n^+$**.**

Further, define $\|X\|_! := \| \, |X| \, \|_!$ **for all** $X \in \mathbb{M}_n$**.**

Symmetric anti-norm version of the von Neumann theorem

A bijective correspondence between

$$\|\cdot\|_! \text{ symmetric anti-norms on } \mathbb{M}_n$$

$$\updownarrow$$

$$\Phi_! : \mathbb{R}_+^n \to [0,\infty) \text{ homogeneous and concave functions invariant under coordinate permutations}$$

is determined by

$$\|A\|_! = \Phi_!(\lambda(A)), \qquad A \in \mathbb{M}_n^+.$$

Examples

- **The Ky Fan k-anti-norm on $\mathbb{M}_n$, i.e., the sum of the k smallest singular values:**

$$\|A\|_{\{k\}} := \sum_{j=1}^{k} \mu_{n+1-j}(A)$$

- **The Schatten q-norm (or q-quasi-norm) for $q \in (0,1)$:**

$$\|A\|_q := \left(\sum_{j=1}^{n} \mu_j^q(A) \right)^{1/q}$$

This $\|A\|_q$ is an anti-norm also for $q < 0$, where $\|A\|_q = 0$ if A is non-invertible.

- **For $k = 1, \ldots, n$, the functional**

$$\Delta_k(A) := \left(\prod_{j=1}^{k} \mu_{n+1-j}(A) \right)^{1/k}$$

is a symmetric anti-norm on $\mathbb{M}_n$. In particular, $\Delta_n(A) = \det^{1/n}|A|$

- **If $A \mapsto \|A\|_!$ is a symmetric anti-norm on $\mathbb{M}_n^+$, then so is also $A \mapsto \|A^q\|_!^{1/q}$ for any $q \in (0,1)$.**

Theorem **Let** $g, f : [0, \infty) \to [0, \infty)$**,** $0 < q < 1 < p$**,** $A,\ B \in \mathbb{M}_n^+$**.**

- **If** $g(t)$ **is convex and** $g^q(t)$ **is subadditive,**

  $$\|g(A+B)\|^q \le \|g(A)\|^q + \|g(B)\|^q$$

  for any symmetric norm.

- **If** $f(t)$ **is concave and** $f^p(t)$ **is superadditive,**

  $$\|f(A+B)\|_!^p \ge \|f(A)\|_!^p + \|f(B)\|_!^p$$

  for any symmetric anti-norms.

Note **Most convex functions** $g : [0, \infty) \to [0, \infty)$ **are not subadditive: if** $g(0) = 0$ **then it is automatically superadditive. So we need to consider** $g^q(t)$ **for some** $q \in (0, 1)$ **to get subadditivity.**

The proof of the theorem is based on majorization and the Ky Fan dominance principle:

- $\|A\|_{(k)} \le \|B\|_{(k)}$ **$(k = 1, \ldots, n)$** $\Longrightarrow \|A\| \le \|B\|$ **for every symmetric norm**
- $\|A\|_{\{k\}} \le \|B\|_{\{k\}}$ **$(k = 1, \ldots, n)$** $\Longrightarrow \|A\|_! \le \|B\|_!$ **for every symmetric anti-norm**

Cor

- **If** $g(t) = a_0 + a_1 t + \cdots + a_m t^m$ **with** $a_k \ge 0$ **then**

$$\|g(A+B)\|^{1/m} \le \|g(A)\|^{1/m} + \|g(B)\|^{1/m}$$

for $A, B \in \mathbb{M}_n^+$ **and any symmetric norm.**

- **If** $f(t) = a_1 t + a_2 t^{1/2} + \cdots + a_m t^{1/m}$ **with** $a_k \ge 0$ **then**

$$\|f(A+B)\|_!^m \ge \|f(A)\|_!^m + \|f(B)\|_!^m$$

for $A, B \in \mathbb{M}_n^+$ **and any symmetric anti-norm.**

Cor **Let** $g, f : [0, \infty) \to [0, \infty)$**,** $0 < q < 1 < p$**,** $A = [a_{ij}] \in \mathbb{M}_n^+$**.**

- **If** $g(t)$ **is convex with** $g(0) = 0$ **and** $g^q(t)$ **is subadditive,**

$$\operatorname{Tr} g(A) \leq \left(\sum_{i=1}^{n} g^q(a_{ii}) \right)^{1/q}.$$

- **If** $f(t)$ **is concave and** $f^p(t)$ **is superadditive,**

$$\operatorname{Tr} f(A) \geq \left(\sum_{i=1}^{m} f^p(a_{ii}) \right)^{1/p}.$$

Note **The above reverse the standard majorization inequalities:**

$$\operatorname{Tr} g(A) \geq \sum_{i=1}^{n} g(a_{ii}), \quad \operatorname{Tr} f(A) \leq \sum_{i=1}^{n} f(a_{ii}).$$

3. Characterization of convexity/concavity of certain trace functions

Let $\tau := (1/n)\mathrm{Tr}$ on $\mathbb{M}_n$, $\varphi : \Lambda \to \mathbb{R}$ and $f : \Omega \to \Lambda$, where Λ, Ω are intervals. When $A \in \mathbb{M}_n\{\Omega\} \mapsto \varphi \circ \tau \circ f(A)$ is convex or concave?

Theorem **Let φ be continuous on Λ, $\Xi := \varphi(\Lambda)$, and assume that φ is C^2 on Λ°, $\varphi'(t) > 0$ and $\varphi''(t) < 0$ (resp., $\varphi''(t) > 0$) on Λ°. TFAE:**

(i) $\varphi'(t)/\varphi''(t)$ **is convex (resp., concave) on Λ°,**

(ii) $(x_1, \dots, x_n) \in \Xi^n \mapsto \varphi\big(\frac{1}{n}\sum_{i=1}^n \varphi^{-1}(x_i)\big)$ **is convex (resp., concave),**

(iii) $A \mapsto \varphi \circ \tau \circ \varphi^{-1}(A)$ **is convex (resp., concave) on $\mathbb{M}_n\{\Xi\}$,**

(iv) **for any Ω and any $f : \Omega \to \Lambda$ with $\varphi \circ f$ convex (resp., concave), $A \mapsto \varphi \circ \tau \circ f(A)$ is convex (resp., concave) on $\mathbb{M}_n\{\Omega\}$.**

Proof Obvious that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iv):

- φ^{-1} is increasing and convex on Ξ (easily shown by (ii)), so $f = \varphi^{-1} \circ \varphi \circ f$ is convex on Ω.

- **Ky Fan majorization**:

$$\lambda\left(\frac{A+B}{2}\right) \prec \frac{\lambda(A)+\lambda(B)}{2},$$

i.e.,

$$\sum_{i=1}^{k} \lambda_i\left(\frac{A+B}{2}\right) \leq \sum_{i=1}^{k} \frac{\lambda_i(A)+\lambda_i(B)}{2}, \quad k = 1, \ldots, n,$$

with equality for $k = n$, where $\lambda(A)$ is the eigenvalue vector of A in decreasing order.

- $$f\left(\lambda\left(\frac{A+B}{2}\right)\right) \prec_w f\left(\frac{\lambda(A)+\lambda(B)}{2}\right) = \varphi^{-1}\circ\varphi\circ f\left(\frac{\lambda(A)+\lambda(B)}{2}\right)$$
$$\le \varphi^{-1}\left(\frac{\varphi\circ f(\lambda(A))+\varphi\circ f(\lambda(B))}{2}\right)$$

- **Hence**

$$\varphi\circ\tau\circ f\left(\frac{A+B}{2}\right) \le \varphi\circ\tau\circ\varphi^{-1}\left(\frac{\varphi\circ f(\lambda(A))+\varphi\circ f(\lambda(B))}{2}\right)$$
$$\le \frac{\varphi\circ\tau\circ f(\lambda(A))+\varphi\circ\tau\circ f(\lambda(B))}{2} \quad \textbf{(by (ii))}$$
$$= \frac{\varphi\circ\tau f(A)+\varphi\circ\tau\circ f(B)}{2}$$

(i) $\Leftrightarrow$ (ii): For any $t_i \in \Lambda^\circ$, $s_i := \varphi(t_i) \in \Xi^\circ$ and $x_i \in \mathbb{R}$, $1 \le i \le n$, since

$$\frac{d^2}{du^2}\,\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\varphi^{-1}(s_i + ux_i)\right)\Bigg|_{u=0}$$
$$= \varphi''\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{\varphi'(t_i)}\right)^2 - \varphi'\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi''(t_i)x_i^2}{\varphi'(t_i)^3}\right),$$

(ii) is equivalent to

$$(*)\qquad \left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{\varphi'(t_i)}\right)^2 \le \frac{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right)}{\varphi''\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right)}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi''(t_i)x_i^2}{\varphi'(t_i)^3}\right).$$

Letting $x_i := \varphi'(t_i)^2/\varphi''(t_i)$ in $(*)$ gives, thanks to $\varphi'(t)/\varphi''(t) < 0$,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi'(t_i)}{\varphi''(t_i)} \ge \frac{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right)}{\varphi''\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right)},$$

which means (i). Conversely, (i) implies $(*)$ as

$$\left(\frac{1}{n}\sum_{i=1}^n \frac{x_i}{\varphi'(t_i)}\right)^2 = \left(\frac{1}{n}\sum_{i=1}^n \frac{\varphi'(t_i)^{1/2}}{\{-\varphi''(t_i)\}^{1/2}} \cdot \frac{\{-\varphi''(t_i)\}^{1/2} x_i}{\varphi'(t_i)^{3/2}}\right)^2$$

$$\le \frac{\varphi'\left(\frac{1}{n}\sum_{i=1}^n t_i\right)}{\varphi''\left(\frac{1}{n}\sum_{i=1}^n t_i\right)} \left(\frac{1}{n}\sum_{i=1}^n \frac{\varphi''(t_i) x_i^2}{\varphi'(t_i)^3}\right)$$

by using the Schwarz inequality.

Examples

- **When $\varphi(t) = \log t$ on $(0, \infty)$, $\varphi'(t) > 0$, $\varphi''(t) < 0$ and $\varphi'(t)/\varphi''(t) = -t$. Hence the pressure function $\log \tau(e^{f(A)})$ (also $\log \mathrm{Tr}\,(e^{f(A)})$) is convex on $\mathbb{M}_n\{\Omega\}$ for any convex $f : \Omega \to \mathbb{R}$.**
- **When $\varphi(t) = e^t$ on $\mathbb{R}$, $\varphi'(t)/\varphi''(t) = 1$. Hence $\det^{1/n} f(A) = \exp \tau(\log f(A))$ is concave on $\mathbb{M}_n\{\Omega\}$ for any concave $f : \Omega \to (0, \infty)$ (extending Minkowski's inequality again).**

4. Minkowski's inequalities in the semifinite von Neumann algebra setting

Notations

- $(\mathcal{M},\tau)$: **a semifinite von Neumann algebra with a faithful normal semifinite trace** τ
- $\widetilde{\mathcal{M}}$: **the set of** τ**-measurable operators**
  (Note: if $\tau(\mathbf{1}) < +\infty$, $\widetilde{\mathcal{M}}$ **is the set of all densely-defined closed operators affiliated with** $\mathcal{M}$; $\widetilde{\mathcal{M}} = B(\mathcal{H})$ **if** $\mathcal{M} = B(\mathcal{H})$**)**
- $\mathcal{M}^+ := \{x \in \mathcal{M} : x \geq 0\}, \quad \widetilde{\mathcal{M}}^+ := \{x \in \widetilde{\mathcal{M}} : x \geq 0\}$
- **The generalized** s**-numbers of** $x \in \widetilde{\mathcal{M}}$ **are**

$$\mu_t(x) := \inf\{s \geq 0 : \tau(e_{(s,\infty)}(|x|)) \leq t\}, \qquad t \in (0,\infty),$$

where $|x| = \int_0^\infty s\, de_s(|x|)$ **is the spectral decomposition of** $|x|$**.**

Definition

- **For $x \in \mathcal{M}^+$ and $t \in (0, \infty)$,**

$$\Delta_t(x) := \lim_{\varepsilon \searrow 0} \exp\left(-\frac{1}{t}\int_0^t \log \mu_s((x+\varepsilon\mathbf{1})^{-1})\, ds\right)$$
$$= \inf_{\varepsilon>0} \exp\left(-\frac{1}{t}\int_0^t \log \mu_s((x+\varepsilon\mathbf{1})^{-1})\, ds\right) \in [0, \infty)$$

  It is clear that if $x, y \in \mathcal{M}^+$ and $y \le x$, then $\Delta_t(y) \le \Delta_t(x)$.

- **For $x \in \widetilde{\mathcal{M}}^+$,**

$$\Delta_t(x) := \sup\{\Delta_t(y) : y \in \mathcal{M}^+,\ y \le x\}$$

- **Further, define $\Delta_t(x) := \Delta_t(|x|)$ for $x \in \widetilde{\mathcal{M}}$.**

Example **When** $(\mathcal{M},\tau)=(\mathbb{M}_n,\mathrm{Tr}\,)$**, for** $A\in\mathbb{M}_n$**,**

$$\Delta_k(A)=\lim_{\varepsilon\searrow 0}\exp\left(-\frac{1}{k}\sum_{j=1}^{k}\log\mu_j((|A|+\varepsilon\mathbf{1})^{-1})\right)$$
$$=\lim_{\varepsilon\searrow 0}\exp\left(-\frac{1}{k}\sum_{j=1}^{k}\log(\mu_{n-j+1}(A)+\varepsilon)^{-1}\right)=\left(\prod_{j=1}^{k}\mu_{n-j+1}(A)\right)^{1/k},$$

which is a symmetric anti-norm on $\mathbb{M}_n$**.**

Properties

(a) **If** $x,y\in\widetilde{\mathcal{M}}^+$ **and** $y\le x$**, then** $\Delta_t(y)\le\Delta_t(x)$**.**

(b) **For** $x\in\mathcal{M}^+$**,**

$$\Delta_t(x)=\begin{cases}\exp\left(-\dfrac{1}{t}\displaystyle\int_0^t\log\mu_s(x^{-1})\,ds\right) & \textbf{if } \ker x=\{0\} \textbf{ and } x^{-1}\in\widetilde{\mathcal{M}},\\ 0 & \textbf{otherwise}.\end{cases}$$

(c) **If $x_n, x \in \mathcal{M}^+$, $x_n \le \alpha \mathbf{1}$ for some $\alpha \in (0,\infty)$ and $x_n \to x$ in the measure topology, then**

$$\Delta_t(x) \ge \limsup_{n\to\infty} \Delta_t(x_n).$$

(d) **If $x_n, x \in \widetilde{\mathcal{M}}^+$, $x_n \ge \delta \mathbf{1}$ for some $\delta \in (0,\infty)$ and $x_n \to x$ in the measure topology, then**

$$\Delta_t(x) \le \liminf_{n\to\infty} \Delta_t(x_n).$$

(e) **If $x \in \widetilde{\mathcal{M}}^+$ and $x_n := \int_0^n s\, de_s(x)$, then**

$$\Delta_t(x) = \lim_{n\to\infty} \Delta_t(x_n).$$

(f) **If $\tau(\mathbf{1}) < +\infty$ and $x \in \mathcal{M}$, then**

$$\Delta_t(x) = \exp\left(\frac{1}{t}\int_{\tau(\mathbf{1})-t}^{\tau(\mathbf{1})} \log \mu_s(x)\, ds\right), \qquad t \in (0, \tau(\mathbf{1})].$$

In particular, if $\tau(\mathbf{1}) = 1$ and $x \in \mathcal{M}$, then $\Delta_1(x)$ is equal to the Fuglede-Kadison determinant $\Delta(x) = \exp\big(\int_0^1 \log \mu_s(x)\, ds\big)$.

(g) **If** $\tau(\mathbf{1}) = +\infty$ **and** $x \in \widetilde{\mathcal{M}}^+$**, then**

$$\Delta_t(x) = \lim_{\varepsilon \searrow 0} \exp\left(-\frac{1}{t} \int_0^t \log \mu_s((x + \varepsilon \mathbf{1})^{-1})\, ds \right).$$

Proposition **For every** $t > 0$**,** $\Delta_t(x)$ **is concave (equivalently, superadditive) on** $\widetilde{\mathcal{M}}^+$**, i.e.,**

$$\Delta_t(x + y) \geq \Delta_t(x) + \Delta_t(y), \qquad x, y \in \widetilde{\mathcal{M}}^+.$$

Fack (1983) and Fack and Kosaki (1986) discussed a different functional $\Lambda_t(x)$ **for** $x \in \widetilde{\mathcal{M}}$**:**

$$\Lambda_t(x) := \exp \int_0^t \log \mu_s(x)\, ds, \qquad t \in (0, \infty),$$

and established:

- **Generalized Weyl inequality**

$$\Lambda_t(xy) \leq \Lambda_t(x)\Lambda_t(y)$$

More generally, if $f : [0,\infty) \to \mathbb{R}$ **is increasing and** $f(e^t)$ **is convex, then** $\int_0^t f(\mu_s(xy))\,ds \leq \int_0^t f(\mu_s(x)\mu_s(y))\,ds$**.**

- **Conjecture due to Grothendieck**

$$\Lambda_t(\mathbf{1} + |x+y|) \leq \Lambda_t(\mathbf{1} + |x|)\Lambda_t(\mathbf{1} + |y|)$$

Difference of our Δ_t**-functional from** Λ_t **is that** Λ_t **is the "product of the** t**th largest** s**-numbers" while** Δ_t **is the "product of the** t**th smallest** s**-numbers". In fact, when** $(\mathcal{M},\tau) = (\mathbb{M}_n, \mathrm{Tr}\,)$ **and** $A \in \mathbb{M}_n$**,**

$$\Lambda_k(A) = \prod_{j=1}^{k} \mu_j(A), \quad \Delta_k(A) = \left\{ \prod_{j=1}^{k} \mu_{n-j+1}(A) \right\}^{1/k}.$$

Another difference is that the definition of Δ_t **contains the normalization** $1/t$ **inside** $\exp$ **while that of** Λ_t **does not.**

Minkowski's inequalities with a convex or concave function hold true in the semifinite von Neumann setting:

Theorem **If** $g : [0,\infty) \to [0,\infty)$ **is convex with** $g(0) = 0$ **then**

$$\Delta_t(g(x+y)) \geq \Delta_t(g(x)) + \Delta_t(g(y))$$

for every $t \in (0,\infty)$ **and every** $x, y \in \widetilde{\mathcal{M}}^+$**.**

Theorem **If** $f : [0,\infty) \to [0,\infty)$ **is a concave continuous function then**

$$\Delta_t\left(f\left(\frac{x+y}{2}\right)\right) \geq \frac{\Delta_t(f(x)) + \Delta_t(f(y))}{2}$$

for every $t \in (0,\infty)$ **and every** $x, y \in \widetilde{\mathcal{M}}^+$**.**

Thank you for your attention.