H_p-THEORY FOR CONTINUOUS FILTRATIONS

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Joint work with Marius JUNGE

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- **Aim:** To study martingales with respect to continuous filtrations in the setting of von Neumann algebras (VNA).

- **Long term goal:** To develop a theory of stochastic integration in VNA.

    - Classical probabilities:stochastic calculus, stochastic integrals, Ito formula, martingales with continuous parameter are well known.
    - Quantum stochastic calculus is also well developed.
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  - VNA setting:theory of discrete martingales is now well developed. Continuous case? Stochastic integration?

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Note that in the classical case the convergence of the stochastic integrals is closely related to the existence of the quadratic variation bracket $[\cdot,\cdot]$. via the formula

$$X_t Y_t = \int^t X_{s^-} dY_s + \int^t Y_{s^-} dX_s + [X, Y]_t,$$

where $[X,Y]_t = X_0 Y_0 + \lim_{n\to\infty} \sum_{0\leq k<2^n} (X_{t\frac{k+1}{2^n}} - X_{t\frac{k}{2^n}})(Y_{t\frac{k+1}{2^n}} - Y_{t\frac{k}{2^n}})$

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We first study this bracket $[\cdot, \cdot]$ and in particular its $L_{p/2}$-norm by considering the Hardy spaces H_p defined in the classical case by

$$\|x\|_{H_p} = \|[x, x]\|_{p/2}^{1/2}.$$

Discrete case

Let $(N_k)_{k\geq 1}$ be a discrete filtration and (E_k) be the associated conditional expectations. Recall that a **discrete noncommutative martingale** is a sequence $x=(x_k)$ in $L_1(N)$ such that $E_k(x_{k+1})=x_k$, for all $k\geq 1$.

Definition (Discrete Hardy spaces)

Let $1\leq p<\infty$.

- H_p^c is the completion of all finite L_p-martingales x for
$$\|x\|_{H_p^c}=\Big\|\Big(\sum_{k\geq 1}|d_kx|^2\Big)^{1/2}\Big\|_p, \quad \text{where} \quad d_kx=x_k-x_{k-1}.$$
- $H_p^r=\{x : x^*\in H_p^c\}$
- $H_p=\begin{cases} H_p^c+H_p^r & \text{if } 1\leq p\leq 2 \\ H_p^c\cap H_p^r & \text{if } 2\leq p<\infty \end{cases}.$

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Known results in the discrete case

Burkholder-Gundy Theorem (Pisier-Xu ('97), Junge-Xu ('03))

$H_p = L_p(N)$ holds with equivalent norms for $1 < p < \infty$.

Theorem (Pisier-Xu('97), Junge-Xu ('03))

Let $1 < p < \infty$. Then H_p^c is complemented in $L_p(N; \ell_2^c)$.
Hence $(H_p^c)^* = H_{p'}^c$ for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Fefferman-Stein duality (Pisier-Xu ('97) for $p = 1$, Junge-Xu ('03))

Let $1 \leq p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$(H_p^c)^* = L_{p'}^c MO,$$

where $L_{p'}^c MO = \{x \in L_2(N) : \|x\|_{L_{p'}^c MO} < \infty\}$ and
$\|x\|_{L_{p'}^c MO} = \|\sup_k^+ E_k|x - x_{k-1}|^2\|_{p'/2}^{1/2}$. (If $p' = \infty$, we denote BMO^c.)

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Definition of $\mathcal{H}_p$-spaces in the continuous case

Let $(N_t)_{t\geq 0}$ be a right continuous filtration and let $1 \leq p < \infty$. We fix an interval $[0, T]$ and consider a finite partition $\sigma = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. Let $\mathcal{U}$ be an ultrafilter refining the general net of partitions given by inclusion.
Then we can define two norms on N:

1. We consider the finite bracket $[x, x]_T^\sigma = \sum_{1\leq k\leq n} |d_{t_k} x|^2$, where $d_{t_k} x = E_{t_k}(x) - E_{t_{k-1}}(x)$. We may simply define

$$[x, x]_T = w^*\text{-}\lim_{\sigma,\mathcal{U}} [x, x]_T^\sigma \quad \text{and} \quad \|x\|_{\hat{\mathcal{H}}_p^c} = \|w^*\text{-}\lim_{\sigma,\mathcal{U}} [x, x]_T^\sigma\|_{p/2}^{1/2}.$$

2. Natural version:

$$\|x\|_{\mathcal{H}_p^c} = \lim_{\sigma,\mathcal{U}} \|[x, x]_T^\sigma\|_{p/2}^{1/2} = \lim_{\sigma,\mathcal{U}} \|x\|_{H_p^c(\sigma)}.$$

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$\Rightarrow$ Taking the corresponding completions we define $\hat{\mathcal{H}}_p^c$ with limit inside and $\mathcal{H}_p^c$ with limit outside.

Main result

Lemma

Let $x \in N$ and $1 \leq p < \infty$. Then

$$\|x\|_{\mathcal{H}_p^c} \approx \begin{cases} \sup_\sigma \|x\|_{H_p^c(\sigma)} & \text{if } 1 \leq p \leq 2 \\ \inf_\sigma \|x\|_{H_p^c(\sigma)} & \text{if } 2 \leq p < \infty \end{cases}$$

Main Theorem

1. *Let $1 \leq p < \infty$. Then $\mathcal{H}_p^c$ is independent of the choice of the ultrafilter $\mathcal{U}$.*
2. *$\mathcal{H}_p^c = \hat{\mathcal{H}}_p^c$ hold with equivalent norms for $1 \leq p < \infty$.*
3. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $(\mathcal{H}_p^c)^* = \mathcal{H}_{p'}^c$.*

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Naive candidate for the $\mathcal{BMO}^c$-norm:

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Definition

The unit ball of $\mathcal{BMO}^c$ is given by the L_2-closure of

$$\{x = \text{w}^*\text{-}\lim_\sigma x(\sigma); \lim_\sigma \|x(\sigma)\|_{BMO^c(\sigma)} \leq 1\}.$$

Burkholder Theorem/Davis' decomposition:Discrete case

Let $(N_k)_{k\geq 1}$ be a discrete filtration.

Definition

Let $1 \leq p < \infty$.

- The conditioned Hardy space h_p^c is the completion of all finite L_∞-martingales x for $\|x\|_{h_p^c} = \left\| \left(\sum_k E_{k-1}(|d_k x|^2) \right)^{1/2} \right\|_p$.
- $h_p^r = \{x : x^* \in h_p^c\}$.
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- The conditioned Hardy space h_p^c is the completion of all finite L_∞-martingales x for $\|x\|_{h_p^c} = \Big\| \Big(\sum_k E_{k-1}(|d_k x|^2) \Big)^{1/2} \Big\|_p$.
- $h_p^r = \{x : x^* \in h_p^c\}$.
- The diagonal Hardy space h_p^d is the subspace of $\ell_p(L_p(N))$ consisting of martingale difference sequences.
- $h_p = \begin{cases} h_p^c + h_p^r + h_p^d & \text{if } 1 \leq p \leq 2 \\ h_p^c \cap h_p^r \cap h_p^d & \text{if } 2 \leq p < \infty \end{cases}$.

Burkholder Theorem/Davis' decomposition:Discrete case

Let $(N_k)_{k\geq 1}$ be a discrete filtration.

Burkholder Theorem (Junge-Xu ('03))

$h_p = L_p(N)$ holds with equivalent norms for $1 < p < \infty$.

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Using a dual approch we can prove that this also holds for $p = 1$ (Junge-Mei, P., '09), and we get a column version of Davis' decomposition:

$$H_p^c = \begin{cases} h_p^c + h_p^d & \text{if } 1 \leq p \leq 2 \\ h_p^c \cap h_p^d & \text{if } 2 \leq p < \infty \end{cases}$$

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$\Rightarrow$ we can actually prove a stronger Davis' decomposition

$$H_p^c = h_p^c + h_p^{1_c} \quad \text{if } 1 \leq p \leq 2,$$

where the spaces $h_p^{1_c}$ have a nice regularity property : $h_p^{1_c} \subset h_q^{1_c}$ for $q \leq p$.

Davis' decomposition:Continuous case

In the continuous situation we define the norms

$$\|x\|_{\mathsf{h}_p^i} = \lim_{\sigma,\mathcal{U}} \|x\|_{h_p^i(\sigma)} \quad \text{for } i = c, d, 1_c \quad \text{and } 1 \leq p < 2.$$

By a nonstandard analysis approach, we first prove results for ultraproduct spaces, then use regularization process, and finally we take the w*-limit (ie the standard part) to get

Theorem

Let $1 \leq p < 2$. *Then*

$$\mathcal{H}_p^c = \mathsf{h}_p^c + \mathsf{h}_p^{1_c} = \mathsf{h}_p^c + \mathsf{h}_p^d.$$

Randrianantoanina's decomposition for conditioned square functions

Let $(N_k)_{1\leq k\leq m}$ be a discrete filtration.

Theorem (Randrianantoanina ('05))

Let $x \in L_2(N)$. Then there exists a decomposition $x = x_1 + x_2 + x_3$ such that $\|x_i\|_2 \leq C\|x\|_2$ and

$$\Big\|\Big(\sum_{k=1}^{m} E_{k-1}|d_k(x_1)|^2\Big)^{1/2}\Big\|_{1,\infty} + \Big\|\Big(\sum_{k=1}^{m} E_{k-1}|d_k(x_2)^*|^2\Big)^{1/2}\Big\|_{1,\infty}$$
$$+\Big\|\sum_{k=1}^{m} e_{k,k}\otimes d_k(x_3)\Big\|_{L_{1,\infty}(B(\ell_2^m)\overline{\otimes}N)} \leq C\|x\|_1.$$

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Corollary

Let $1 < p < 2$ and $x \in L_2(N)$. Then there exists a decomposition $x = x_1 + x_2 + x_3$ such that $\|x_i\|_2 \leq f_p(\|x\|_2, \|x\|_p)$ and

$$\|x_1\|_{h_p^c} + \|x_2\|_{h_p^r} + \|x_3\|_{h_p^d} \leq C(p)\|x\|_p.$$

Randrianantoanina's decomposition for conditioned square functions:column version

Corollary

Let $(N_k)_{1\leq k\leq m}$ *be a discrete filtration. Let* $1 < p < 8/7$ *and* $x \in L_2(N)$*. Then there exists a decomposition* $x = x_1 + x_2$ *such that* $\|x_i\|_2 \leq f_p(\|x\|_2, \|x\|_p)$ *and*

$$\|x_1\|_{h_p^c} + \|x_2\|_{h_p^d} \leq C(p)\|x\|_{H_p^c}.$$

Theorem (Continuous case)

Let $1 < p < 2$*. Then* $\mathcal{H}_p^c \overset{C(p)}{=} \mathrm{h}_p^c \diamond \mathrm{h}_p^d$*, where* $\mathrm{h}_p^c \diamond \mathrm{h}_p^d$ *is the completion of* $L_2(N)$ *with respect to the norm*

$$|||x|||_p = \inf \|x_1\|_{\mathrm{h}_p^c} + \|x_2\|_{\mathrm{h}_p^d},$$

where the infimum runs over all decomposition $x = x_1 + x_2$ *with* $x_1 \in L_2(N), x_2 \in L_2(N) \cap \mathrm{h}_p^d$*.*

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Fefferman-Stein duality

We don't know how to describe the dual space of h_p^d.
Observation: $\mathrm{h}_p^c \diamond \mathrm{h}_p^d = \mathrm{h}_p^c \diamond K_p(t; \mathrm{h}_p^d, L_2(N))$ for all $t \geq 1$, where $K_p(t; L_2(N), \mathrm{h}_p^d)$ is the completion of L_2 with respect to the norm

$$\|x\|_{K_p(t;\mathrm{h}_p^d, L_2(N))} = \inf \|x_1\|_{\mathrm{h}_p^d} + t\|x_2\|_2,$$

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Theorem

Let $1 \leq p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$B_{(\mathcal{H}_p^c)^*} = \{x \in L_2 : \|x\|_{L^c_{p'}\mathrm{mo}} \leq 1,\ x = L_2\text{-}\lim_\lambda x_\lambda,\ \|x_\lambda\|_2 \leq K, \|x_\lambda\|_{\mathsf{h}^d_{p'}} \leq 1\}$$

with equivalent norms.

Further results:

- Complementation of $\mathcal{H}_p$ spaces in larger ultraproduct spaces, which have L_p-module structures.
- Interpolation of $\mathcal{H}_p$ spaces.
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APPLICATIONS AND PERSPECTIVES : Interpolation of semigroups, convergence of stochastic integrals in the setting of VNA.

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THANK YOU FOR YOUR ATTENTION