Subproduct systems

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Introduction

What is a subproduct system?

But first of all, **WHY?**

1. B. Solel and I were led to consider subproduct systems when trying prove a **multiparameter** version of Bhat's Theorem.
2. B. V. R. Bhat and M. Mukherjee were led to consider subproduct systems to faciltate computations in amalgamated product systems (they called them **inclusion systems**).
3. Subproduct systems have also been considered in the theory of non-selfadjoint operator algebras.

(In fact, subproduct systems were always there: Arveson, Bhat, Bhat-Bhattacharyya, Bhat-Skeide, Dey, Markiewicz, Muhly-Solel and surely others too).

What is a subproduct system?

$\mathbb{S}$ – a semigroup (take $\mathbb{S} \subseteq \mathbb{R}_+^k$).
A **product system** is a family $\{X(s)\}_{s \in \mathbb{S}}$ of Hilbert C*-correspondences such that

$$X(s+t) = X(s) \otimes X(t)$$

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$$X(s+t) \subseteq X(s) \otimes X(t)$$

Recall: W*-correspondences

$\mathcal{M}$ a v.N. algebra.
A W*-correspondence E over $\mathcal{M}$ is a right Hilbert C* module over $\mathcal{M}$ with additional structure:

1. A left action by adjointable operators: $\langle x, ay\rangle = \langle a^*x, y\rangle$.
2. E is **self dual**.

If F is another correspondence over $\mathcal{M}$, we define $E \otimes F$ to be the algebraic tensor product with natrual Hilbert module operations:

$$a(x \otimes y)b = ax \otimes yb \quad , \quad x \in E, y \in F, a, b \in \mathcal{M}$$

and

$$\langle x \otimes y, z \otimes w\rangle = \langle y, \langle x, z\rangle\, w\rangle$$

Subproduct systems (in the W* case)

Let $\mathcal{M}$ be a von Neumann algebra.
A **subproduct system** (of Hilbert correspondences over $\mathcal{M}$) is a family $\{X(s)\}_{s\in\mathbb{S}}$ such that

1. $X(0) = \mathcal{M}$
2. There are coisometries $U_{s,t} : X(s) \otimes X(t) \to X(s+t)$.
   ($U_{s,t}$ is adjointable and $U_{s,t}^*$ is an isometric, bimodule map)
3. The multiplication

$$(x, y) \mapsto xy := U_{s,t}(x \otimes y) \quad , \quad x \in X(s), y \in X(t)$$

gives $\bigcup_{s\in\mathbb{S}} X(s)$ the structure of an (associative) semigroup.

Examples

1. Product systems.
2. If E is a Hilbert space, then $\{E^{\otimes n}\}_{n\in\mathbb{N}}$ is a (sub)product system:
$$E^{\otimes m} \otimes E^{\otimes n} = E^{\otimes (m+n)}$$
3. Symmetric tensor powers: $\{E^n\}_{n\in\mathbb{N}}$. Multiplication: for $x \in E^m, y \in E^n$,
$$xy := p_{m+n}(x \otimes y)$$
Certainly:
$$E^{m+n} \subseteq E^m \otimes E^n$$
It is well known that this product is associative.

Chapter 1

How do subproduct systems arise?

Subproduct systems arise naturally in the context of CP-semigroups. There are two (dual) ways this happens:

1. The GNS SPS.
2. The Arveson-Stinespring SPS.

(there are also other routes which we shall not discuss)

Setting the stage

Let H be a Hilbert space. Let $\mathbb{S} \subseteq \mathbb{R}_+^k$ be some semigroup. Let $\mathcal{M} \subseteq B(H)$ be a v.N. algebra.

- **CP-semigroup** - A family $\Theta = \{\Theta_s\}_{s\in\mathbb{S}}$ of normal, completely positive maps on $\mathcal{M}$, satisfying $\Theta_s \circ \Theta_t = \Theta_{s+t}$ and $\Theta_0 = \mathbf{id}$.
- Θ is said to be a **Markov semigroup** (or a CP_0-semigroup) when $\Theta_s(I) = I$ for all s.

Now fix $\mathcal{M}$ and Θ.

The GNS SPS

The GNS SPS - Units

Let $X = \{X(s)\}_{s\in\mathbb{S}}$ be a SPS. A **unit** is a family $\xi = \{x_s\}_{s\in\mathbb{S}}$ with $\xi_s \in X(s)$ such that

$$\xi_s \cdot \xi_t = \xi_{s+t}$$

Unital: $\langle \xi_s, \xi_s \rangle = 1$ for all s.
Contractive: $\langle \xi_s, \xi_s \rangle \leq 1$ for all s.
Defining for $a \in \mathcal{M}$

$$\Phi_s : a \mapsto \langle \xi_s, a\xi_s \rangle$$

gives rise to a CP-semigroup:

$$\begin{aligned}\Phi_s(\Phi_t(a)) &= \langle \xi_s, \langle \xi_t, a\xi_t \rangle \, \xi_t \rangle = \langle \xi_t \otimes \xi_s, a\xi_t \otimes \xi_s \rangle = \langle \xi_t \cdot \xi_s, a\xi_t \cdot \xi_s \rangle \\ &= \langle \xi_{s+t}, a\xi_{s+t} \rangle = \Phi_{s+t}(a)\end{aligned}$$

The GNS SPS

.... and every CP-semigroup arises this way

Theorem (Paschke, Bhat-Skeide)

For every CP-semigroup Θ *on* $\mathcal{M}$ *there is a SPS* $F_\Theta = \{F_\Theta(s)\}_{s\in\mathbb{S}}$ *of* $\mathcal{M}$*-correspondences and a unit* $\xi_\Theta = \{\xi_s\}_{s\in\mathbb{S}}$ *in* X *such that*

$$\Theta_s(a) = \langle \xi_s, a\xi_s \rangle .$$

Remarks:

- If Θ is an E-semigroup, then F_Θ is a product system.
- Θ is unit preserving (contractive) if and only if ξ is a unital (contractive) unit.

Construction of the GNS SPS

1. For each fixed $s \in \mathbb{S}$ we let $(F_\Theta(s), \xi_s)$ be the GNS representation of Θ_s:

$$F_\Theta(s) = \mathcal{M} \otimes_{\Theta_s} \mathcal{M} \quad , \quad \xi_s = 1 \otimes 1$$

   inner product: $\langle a \otimes b, c \otimes d \rangle = b^* \Theta_s(a^* c) d$, so

$$\langle \xi_s, a\xi_s \rangle = \langle 1 \otimes 1, a \otimes 1 \rangle = 1^* \Theta_s(a) 1 = \Theta_s(a)$$

2. Define an isometry $V_{s,t} : F_\Theta(s+t) \to F_\Theta(s) \otimes F_\Theta(t)$ by

$$V_{s,t} : \xi_{s+t} \mapsto \xi_s \otimes \xi_t$$

3. Define $U_{s,t} = V_{s,t}^*$.

The maps $U_{s,t}$ induce the required multiplication on $F_\Theta = \{F_\Theta(s)\}$, $\xi_\Theta = \{\xi_s\}_{s \in \mathbb{S}}$ is a unit, and (F_Θ, ξ_Θ) does the job.

Minimality and uniqueness of the GNS SPS

Theorem (S.-Skeide)

Let Θ be a CP-semigroup on $\mathcal{M}$. If $X = \{X(s)\}_{s \in \mathbb{S}}$ is a SPS of $\mathcal{M}$-correspondences and $\eta = \{\eta_s\}_{s \in \mathbb{S}}$ is a unit in X such that

$$\Theta_s(a) = \langle \eta_s, a\eta_s \rangle .$$

Then F_Θ can be identified with a **subproduct subsystem** *of X, so that ξ_Θ is identified with η. If η generates X, then $F_\Theta = X$.*

The Arveson-Stinespring SPS

The Arveson-Stinespring SPS - Representations

Subproduct system representations

A family $T = \{T_s\}_{s \in \mathcal{S}}$ is called a representation of $X = \{X(s)\}_{s \in \mathbb{S}}$ if for all s, $T_s : X(s) \to B(H)$ is a completely bounded map such that

$$T_s(axb) = aT_s(x)b,$$

and

$$T_{s+t}(x \cdot x') = T_s(x)T_t(x')$$

for $x \in X(s)$ and $x' \in X(t)$. In other words

$$T_{s+t}(U_{s,t}(x \otimes x')) = T_s(x)T_t(x').$$

We define $\widetilde{T}_s : X(s) \otimes H \to H$ by $\widetilde{T}_s(\xi \otimes h) = T_s(\xi)h$.

The AS SPS

Theorem (Arveson, Muhly-Solel)

Let $\Theta = \{\Theta_s\}_{s\in\mathbb{S}}$ be a CP-semigroup on $M \subseteq B(H)$. Then there exist a SPS $E_\Theta = \{E_\Theta(s)\}_{s\in\mathbb{S}}$ of $\mathcal{M}'$-correspondences and representation T of Θ on H such that

$$\Theta_s(a) = \widetilde{T}_s \left(I_{E_\Theta(s)} \otimes a\right) \widetilde{T}_s^*.$$

Remarks:

- If Θ is an E-semigroup, then X is a product system, and $\widetilde{T}_s$ is an isometry for all s.
- If $\widetilde{T}_s$ is an isometry for all s, then Θ is an E-semigroup.
- Θ is unit preserving (contractive) if and only if $\widetilde{T}_s$ is a coisometry (a contraction) for all s.
- **Every** subproduct system arises this way.

Construction of the AS SPS

1. $\mathcal{M} \otimes_{\Theta_s} H$ with inner product $\langle a \otimes g, b \otimes h \rangle = \langle g, \Theta_s(a^* b) h \rangle$.
2. $W_{\Theta_s} : H \to \mathcal{M} \otimes_{\Theta_s} H$, given by $W_{\Theta_s}(h) = I \otimes h$.
3. Construct the W*-correspondence over $\mathcal{M}'$:

$$E_\Theta(s) = \{\xi \in B(H, \mathcal{M} \otimes_{\Theta_s} H) : \forall a \in \mathcal{M}. \xi a = (a \otimes I)\xi\}.$$

4. $T_s : E_\Theta(s) \to B(H)$ given by $T_s(\xi) = W_{\Theta_s}^* \xi$.
5. $$\Theta_s(a) = \widetilde{T}_s \left(I_{E_\Theta(s)} \otimes a \right) \widetilde{T}_s^* .$$

Fact:

$E_\Theta = \{E_\Theta(s)\}_{s \in \mathbb{S}}$ is a subproduct system, and $T = \{T_s\}_{s \in \mathbb{S}}$ is a representation.

Important case: $\mathcal{M} = B(H)$

Then $\mathcal{M}' = \mathbb{C}$, $E_\Theta(s)$ is a Hilbert space. If $\dim E_\Theta(s) = n$, then $E_\Theta(s) \otimes H = H \oplus \ldots \oplus H$.
Fixing s, $\widetilde{T}_s$ is a row operator

$$\widetilde{T}_s = (t_1, \ldots, t_n) \quad , \quad t_i \in B(H)$$

(Recall: $\widetilde{T}_s : E_\Theta(s) \otimes H \to H$ is defined by $\widetilde{T}_s(x \otimes h) = T_s(x)h$).
We recover the "Krauss Decomposition" of Θ_s:

$$\Theta_s(a) = \widetilde{T}_s \left(I_{E_\Theta(s)} \otimes a \right) \widetilde{T}_s^* = \sum t_k a t_k^*.$$

Minimality and uniqueness of the AS representation

Theorem (S.-Solel)

Let Θ be a CP-semigroup on $\mathcal{M}$ and let (E_Θ, T) be the Arveson-Stinespring representation of Θ:

$$\Theta_s(a) = \widetilde{T}_s \left(I_{E_\Theta(s)} \otimes a \right) \widetilde{T}_s^* .$$

If X is another subproduct system and $R : X \to B(H)$ is a representation such that

$$\Theta_s(a) = \widetilde{R}_s \left(I_{X(s)} \otimes a \right) \widetilde{R}_s^* ,$$

then E_Θ is a **subproduct subsystem** *of X, and $R\big|_{E_\Theta} = T$.*
If R is injective, then $X = E_\Theta$ and $R = T$.

Chapter 2

Subproduct systems in dilation theory

E-dilation of CP-semigroups

Given a semigroup $\Theta = \{\Theta_s\}_{s \in \mathbb{S}}$ of CP maps acting on $\mathcal{M} \subseteq B(H)$, an **E-dilation** of Θ is a triple $(\alpha, \mathcal{R}, K)$, where

- $K \supseteq H$ is a Hilbert space,
- $\mathcal{R} \subseteq B(K)$ is a v.N. algebra such that $\mathcal{M} = P_H \mathcal{R} P_H$,
- $\alpha = \{\alpha_s\}_{s \in \mathbb{S}}$ is an E-semigroup such that

$$\Theta_s(P_H T P_H) = P_H \alpha_s(T) P_H,$$

  for all $T \in \mathcal{R}, s \in \mathbb{S}$. In particular, for $T \in \mathcal{M}$,

$$\Theta_s(T) = P_H \alpha_s(T) P_H.$$

- If Θ is unit preserving we would like α to be unital as well.

We will consider only **contractive** CP-semigroups.

Bhat's Theorem

Theorem (Bhat, SeLegue, Bhat-Skeide, Muhly-Solel, Arveson)

Let $\Theta = \{\Theta_t\}_{t \geq 0}$ be a CP-semigroup acting on $\mathcal{M} \subseteq B(H)$. Then Θ has an E-dilation.

That is, there is:

- *a Hilbert space $K \supseteq H$*
- *a v.N. algebra $\mathcal{R} \subseteq B(K)$ such that $\mathcal{M} = P_H \mathcal{R} P_H$*
- *an E-semigroup $\{\alpha_t\}_{t \geq 0}$ on $\mathcal{R}$ such that for all $T \in \mathcal{R}, t \geq 0$*

$$\Theta_t(P_H T P_H) = P_H \alpha_t(T) P_H.$$

Problem

Can similar results be obtained if $\mathbb{R}_+$ is replaced by some other semigroup $\mathbb{S}$ (say $\mathbb{S} \subseteq \mathbb{R}_+^k$)?

- The case $\mathbb{S} = \mathbb{N}$ corresponds to a single CP map (OK).
- The case $\mathbb{S} = \mathbb{N}^k$ corresponds to k commuting CP maps. $k = 2$ was solved by Bhat (for $\mathcal{M} = B(H)$) and later by Solel (general $\mathcal{M}$).
- The case $\mathbb{S} = \mathbb{R}_+^k$ corresponds to k commuting CP-semigroups. For $\mathbb{S} = \mathbb{R}_+^2$, under an underlying assumption of **strong commutativity**, I showed that every CP_0-semigroup has an E_0-dilation, and that every CP-semigroup on $\mathcal{M} = B(H)$ has an E-dilation.

Main Theorem

Theorem (S.-Solel, S.-Skeide)

A necessary condition for a CP-semigroup Θ to have an E-dilation is that its AS/GNS subproduct system can be embedded as a **subproduct subsystem** *of a product system. This condition is not sufficient.*
If Θ is unit preserving, then embeddability of the AS/GNS subproduct system in a product system is a sufficient condition for the existence of a (unit preserving) E-dilation.

Remarks:

- Both theorems (AS/GNS) were proved separately, but they imply one another.
- The GNS construction works for semigroups on C^*-algebras, and the theorem holds (more-or-less) in that category as well.

Applications

Let's see some applications of this theorem to dilation theory. We will illustrate two kinds of applications:

- Existence results.
- Nonexistence results.

Theorem (Bhat, Solel)

Every pair of commuting CP maps has an E-dilation. That is, every (contractive) CP-semigroup over $\mathbb{N}^2$ has an E-dilation.

This means, if Θ and Φ are commuting CP maps on $\mathcal{M} \subseteq B(H)$, there is

- a Hilbert space $K \supset H$
- a v.N. algebra $\mathcal{R} \subseteq B(K)$ with $\mathcal{M} = P_H \mathcal{R} P_H$
- two commuting $*$-endomorphisms α, β on $\mathcal{R}$ such that

$$\Theta^m(\Phi^n(P_H T P_H)) = P_H \alpha^m(\beta^n(T)) P_H \quad , \quad m, n \in \mathbb{N}$$

Question: What about **three** commuting CP maps?

Nonexistence of dilations

Facts:

- There exists a subproduct system over $\mathbb{N}^3$ that cannot be embedded in any product system (S.-Solel, hint from E. Levy).
- This subproduct system is the AS SPS of some CP-semigroup over $\mathbb{N}^3$.
- **Recall:** If a CP-semigroup has an E-dilation then (by the "Main Theorem") its AS SPS must be embeddable into a product system.

Corollary (S.-Solel)

There exists a CP-seigroup over $\mathbb{N}^3$ that has no E-dilation.

Question: What conditions are sufficient to gaurantee the existence of a dilation for a k-tuple of commuting CP maps?

Guess (well founded): unit preserving?

Unitalization

Let Θ be a CP semigroup acting on $B(H)$.
Define a semigroup $\tilde{\Theta}$ of **unital** CP maps on $B(H \oplus \mathbb{C})$ by

$$\tilde{\Theta}_s \begin{pmatrix} A & h \\ g^* & c \end{pmatrix} = \begin{pmatrix} \Theta_s(A) + c(I - \Theta_s(I)) & 0 \\ 0 & c \end{pmatrix}$$

Theorem (S.-Skeide)

If $\tilde{\Theta}$ has an E-dilation, then Θ has an E-dilation.

Nonexistence of dilations - unital example

Take the example of CP-semigroup $\Theta = \{\Theta_n\}_{n \in \mathbb{N}^3}$ over $\mathbb{N}^3$ that has no E-dilation. Let $\tilde{\Theta}$ be its unitalization. If $\tilde{\Theta}$ had an E-dilation, then Θ would have an E-dilation.

Corollary (S.-Skeide)

There exists a **Markov** *semigroup over* $\mathbb{N}^3$ *with no E-dilation.*

An example exists on $\mathcal{M} = M_k(\mathbb{C})$ with $k = 6$.

Challenge: What is the minimal dimension k for which such a counter example exists?

Recall this result:

Theorem (Bhat, Solel)

Every pair of commuting CP maps has an E-dilation.

This means, if Θ and Φ are commuting CP maps on $\mathcal{M} \subseteq B(H)$, there is a Hilbert space $K \supset H$, a v.N. algebra $\mathcal{R} \subseteq B(K)$ and commuting $*$-endomorphisms α, β on $\mathcal{R}$ such that

$$\Theta^m(\Phi^n(P_H T P_H)) = P_H \alpha^m(\beta^n(T)) P_H \quad , \quad m, n \in \mathbb{N}$$

Remark: One would like to know that if Θ, Φ are unit preserving, then α, β are also unit preserving.

Existence of unital dilations

Theorem (S.-Skeide)

Every pair of commuting unit preserving CP maps has a unit preserving E-dilation.

Proof.

Let $\Theta = \{\Theta_s\}_{s \in \mathbb{N}^2}$ be a Markov semigroup over $\mathbb{N}^2$. By the "Main Theorem", all one has to do is embedd the GNS SPS of Θ in a product system over $\mathbb{N}^2$. The semigroup $\mathbb{N}^2$ is *just* simple enough so that we can pull this off. □

Challenge: Prove this theorem in the C^*-category.

Unitalization revisited:

Let Θ be a CP semigroup acting on $\mathcal{M}$.
Define a semigroup $\tilde{\Theta}$ of **unital** CP maps on $\mathcal{M} \oplus \mathbb{C}$ by

$$\tilde{\Theta}_s(A \oplus c) = (\Theta_s(A) + c(I - \Theta_s(I)), c)$$

Theorem (S.-Skeide)

If $\tilde{\Theta}$ has an E-dilation, then Θ has an E-dilation.

Putting together this result and the unit-preserving dilation result, we recapture Bhat's and Solel's dilation theorem for pairs of commuting CP maps.

Chapter 3

Subproduct systems and operator algebras

The operator algebra $\mathcal{A}_X$ associated with a SPS

- $X = \{X(n)\}_{n\in\mathbb{N}}$ - SPS of finite dimensional Hilbert spaces. Here $X(0) = \mathbb{C}$.
- Let $\{e_1, \dots, e_d\}$ be a basis for $X(1)$.
- Define the X-Fock space

$$\mathcal{F}_X = \oplus_{n=0}^{\infty} X(n)$$

Define the creation operators $S_1, \dots, S_d$ on $\mathcal{F}_X$ by

$$S_i(x) = e_i \cdot x = U_{1,n}(e_i \otimes x) \quad , \quad x \in X(n)$$

Definition

$\mathcal{A}_X$ is the norm closed algebra generated by $S_1, \dots, S_d, 1$.

Examples

1. If $X(n) = E^{\otimes n}$ all n, $\mathcal{A}_X$ is Popescu's noncommutative disc algebra, denoted $\mathcal{T}_+(E)$ by Muhly-Solel.
2. In particular, if $X(n) = \mathbb{C}$ for all n then $\mathcal{A}_X = A(\mathbb{D})$ - the classical disc algebra.
3. If $X(n) = E^n$ (symmetric tensor product) then $\mathcal{A}_X$ is the algebra of continuous multipliers on Drury-Arveson space.

Polynomials and ideals

Let $\mathbb{C}\langle z_1, \dots, z_d \rangle$ be the algebra of polynomials in d non-commuting variables.

A polynomials is **homogeneous** if all its terms have the same total degree

$$p(z) = z_1 z_2 z_3 - 7 z_2 z_3^2 + z_3 z_2 z_3$$

An ideal $\mathcal{I} \subseteq \mathbb{C}\langle z_1, \dots, z_d \rangle$ is said to be **homogeneous** if it is generated by homogeneous polynomials:

$$\mathcal{I} = \langle p_1, \dots, p_k \rangle$$

$\mathcal{A}_X$ is a universal algebra

Fix a Hilbert space E and an o.n.b. $\{e_1, \ldots, e_d\}$.

Theorem (S.-Solel)

There is a bijective correspondence $X \leftrightarrow \mathcal{I}^X$ between SPSs X with $X(1) \subseteq E$ and homogeneous ideals in $\mathbb{C}\langle z_1, \ldots, z_d \rangle$.

Theorem (Popescu)

$\mathcal{A}_X$ is the universal unital operator algebra generated by a row contraction that satisfies the relations in $\mathcal{I}^X$.

Example: if $X(n) = E^n$ (symmetric product) then $\mathcal{I}^X = \langle z_i z_j - z_j z_i \rangle$ is the commutator ideal and $\mathcal{A}_X$ is the universal commutative operator algebra generated by a row contraction.

$*$ $(T_1, \ldots, T_d)$ is a row contraction iff $\sum_{i=1}^d T_i T_i^* \leq I$.

Isomorphic algebras and subproduct systems

Let X and Y be SPSs over $\mathbb{N}$. We say that they are isomorphic ($X \cong Y$) if there is a family of unitaries $V_n : X(n) \to Y(n)$ that respects the product:

$$V_m(x) \cdot V_n(x') = V_{m+n}(x \cdot x')$$

or, more precisely,

$$U^Y_{m,n}(V_m(x) \otimes V_n(x')) = V_{m+n}(U^X_{m,n}(x \otimes x'))$$

Clearly $X \cong Y$ implies that $\mathcal{A}_X$ and $\mathcal{A}_Y$ are unitarily equivalent.
What about the other direction?

Classifying the algebras $\mathcal{A}_X$ with subproduct systems

Theorem (S.-Solel, Davidson-Ramsey-S.)

$X \cong Y$ if and only if $\mathcal{A}_X$ and $\mathcal{A}_Y$ are isometrically isomorphic.

Corollary (Rigidity of the algebras $\mathcal{A}_X$)

If $\mathcal{A}_X$ and $\mathcal{A}_Y$ are isometrically isomorphic then they are unitarily equivalent.

Challenges:

1. General W*-correspondence fibers $X(n)$.
2. SPS over more general semigroups $\mathbb{S}$.

Polynomials and ideals - again

From now on we concentrate on **commutative** SPSs. Let $\mathbb{C}[z_1, \ldots, z_d]$ be the algebra of polynomials in d (commuting) variables.

Theorem

There is a bijective correspondence $X \leftrightarrow \mathcal{I}^X$ between **commutative** *SPSs X with $X(1) \subseteq E$ and homogeneous ideals in $\mathbb{C}[z_1, \ldots, z_d]$.*

For $\mathcal{I} \subseteq \mathbb{C}[z_1, \ldots, z_d]$ and $X \subseteq \mathbb{C}^d$ denote:

$$V(\mathcal{I}) = \{z \in \mathbb{C}^d : \forall p \in \mathcal{I}. p(z) = 0\}$$

$$\mathcal{I}(X) = \{p \in \mathbb{C}[z_1, \ldots, z_d] : \forall x \in X. p(x) = 0\}.$$

An ideal $\mathcal{I}$ is **radical** if $\mathcal{I}(V(\mathcal{I})) = \mathcal{I}$.

Basic problem

How does the geometry of $V(\mathcal{I}^X)$ determine the algebraic and isometric structure of $\mathcal{A}_X$?

Just as in classical algebraic geometry, we will need to assume that the ideals are **radical** to obtain a strong connection between algebra and geometry.

Classifying the algebras $\mathcal{A}_X$ with varieties

Let X and Y be commutative SPSs with finite dimensional fibers. Assume that $\mathcal{I}^X$ and $\mathcal{I}^Y$ are radical.

Theorem (Davidson-Ramsey-S.)

$\mathcal{A}_X$ and $\mathcal{A}_Y$ are isometrically isomorphic if and only if there is unitary U on $\mathbb{C}^d$ such that

$$U(V(\mathcal{I}^X) \cap \mathbb{B}_d) = V(\mathcal{I}^Y) \cap \mathbb{B}_d.$$

$\mathcal{A}_X$ and $\mathcal{A}_Y$ are isomorphic (as algebras) if and only if there is a linear map T on $\mathbb{C}^d$ such that

$$T(V(\mathcal{I}^X) \cap \mathbb{B}_d) = V(\mathcal{I}^Y) \cap \mathbb{B}_d.$$

Caveat: The "isomorphism" part is true under some additional technical condition (works for smooth varieties, irreducible varieties, two irreducible components, hyper-surfaces, dim ≤ 3).

Comparing to classical results

$\mathcal{A}_X$ and $\mathcal{A}_Y$ are isomorphic if and only if there exists a linear map T on $\mathbb{C}^d$ such that

$$T(V(\mathcal{I}^X) \cap \mathbb{B}_d) = V(\mathcal{I}^Y) \cap \mathbb{B}_d.$$

Compare:

$\mathbb{C}[V(\mathcal{I}^X)]$ and $\mathbb{C}[V(\mathcal{I}^Y)]$ are isomorphic if and only if there exists a linear map A on $\mathbb{C}^d$ such that

$$T(V(\mathcal{I}^X)) = V(\mathcal{I}^Y).$$

- Note the difference in "geometry".

Examples

Consider $\mathcal{A}_X$ and $\mathcal{A}_X$, where $V(\mathcal{I}^X)$ and $V(\mathcal{J}^X)$ is a pair of homogeneous varieties in $\mathbb{C}^2$, in one of the following categories

- One line - always isometrically isomorphic.
- Two lines - always isomorphic. Isometrically isomorphic iff angle is the same.
- Three lines - Even the non-closed algebras might be not isomorphic. When they are, sometimes the closed algebras are isomorphic and sometimes not.

However, the C^*-algebra generated by $\mathcal{A}_X$ depends only on the **number** of lines (which is in this case the **topology** of $V(\mathcal{I}^X)$).

Rigidity of varieties and operator algebras

Lemma (Davidson-Ramsey-S.)

Let V be an irreducible variety, and let A be a linear map such that

$$\|Ax\| = \|x\| \ , \ x \in V$$

Then A is isometric on the span of V.

Theorem (Davidson-Ramsey-S.)

Let X and Y be commutative SPS with radical ideals. Assume that $V(\mathcal{I}^X)$ is either **(a)** *irreducible, or* **(b)** *a hypersurface. If $\mathcal{A}_X$ and $\mathcal{A}_Y$ are isomorphic then they are isometrically isomorphic, and therefore also unitarily equivalent.*