Central Limit Theorems for Anticommuting Tensor Independence

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2-graded Algebras

$$\mathfrak{A} = \mathfrak{A}^0 \oplus \mathfrak{A}^1$$
$$\mathfrak{A}^i\mathfrak{A}^j \subset \mathfrak{A}^k \qquad\qquad k \equiv i + j \mod 2$$

The elements of $\mathfrak{A}^0$ are called *even*, those of $\mathfrak{A}^1$ are *odd*. Examples.

- . Trivial gradiation.

$$\mathfrak{A}^0 = \mathfrak{A} \qquad\qquad \mathfrak{A}^1 = \{0\}$$

- . Commutative polynomial algebra $\mathbb{C}[X_i, i \in I]$. The even (odd) elements are the homogeneous polynomials of even (odd) degree and their sums.

- Grassman algebra $\wedge(X_i, i \in I)$. Similar.

- Free algebra $\mathbb{C}\langle X_i, i \in I\rangle$. Similar.

- Clifford algebra generated by $X_i, i \in I$ with defining relations $X_iX_j + X_jX_i = Q_{ij} \in \mathbb{C}$. Similar.

Tensor product $\mathfrak{A}_i, i = 1, \cdots, n$ 2-graded algebras.
The *2-graded tensor product*

$$\bigotimes_{i\in I}^{g} \mathfrak{A}_i = \mathfrak{A}_1 \otimes^g \cdots \otimes^g \mathfrak{A}_n$$

has the underlying vector space

$$\bigotimes_{i\in I} \mathfrak{A}_i = \mathfrak{A}_1 \otimes \cdots \otimes \mathfrak{A}_n.$$

If

$x_i \in \mathfrak{A}^{\alpha_i}$ $\qquad\qquad$ $y_i \in \mathfrak{A}^{\beta_i}$

then

$$\bigotimes_i x_i \bigotimes_i y_i = (-1)^{\sum_{i>j} \alpha_i \beta_j} \bigotimes_i x_i y_i$$

and

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n$$
$$= (-1)^{\alpha_i \alpha_{i+1}} x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n$$

2-graded Commutator

$x \in \mathfrak{A}^{\alpha}, y \in \mathfrak{A}^{\beta}$ $\qquad\qquad$ $[x,y]^g = xy - (-1)^{\alpha\beta} yx$

Anticommutative Algebras

$$x, y \in \mathfrak{A} \Longrightarrow [x,y]^g = 0$$

Examples.

- . Commutative polynomial algebra with trivial gradiation.

- Grassmann algebra.

Let $\mathfrak{A}, \mathfrak{B}$ two 2-graded algebras. A linear mapping $\mathfrak{A} \to \mathfrak{B}$ is called *even*, if it maps $\mathfrak{A}^i \to \mathfrak{B}^i$ for $i = 0, 1$.

Free Algebras

Let $\mathfrak{F} = \mathbb{C}\langle X_i, i \in I\rangle$ be a free algebra and $\mathfrak{A}$ an algebra. A mapping $X_i \mapsto a_i \in \mathfrak{A}$ extends to a homomorphism $\eta : \mathfrak{F} \to \mathfrak{A}$ by

$$\eta(X_{i_1} \cdots X_{i_k}) = a_{i_1} \cdots a_{i_k}$$

.

$\mathfrak{F}$ is a bialgebra.
Coproduct Δ and counit δ

$$\begin{aligned} \Delta &: \mathfrak{F} \to \mathfrak{F} \otimes^g \mathfrak{F} \\ &X_i \mapsto X_i \otimes 1 + 1 \otimes X_i \qquad\qquad 1 \mapsto 1 \otimes 1 \\ \delta &: \mathfrak{F} \to \mathbb{C} \\ &X_i \mapsto 0 \qquad\qquad 1 \mapsto 1 \end{aligned}$$

Iterated coproduct

$$\Delta_3 : \mathfrak{F} \to \mathfrak{F}^{\otimes^g 3}$$
$$\Delta_3 = (\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$$
$$\Delta_3 X_i = X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 + 1 \otimes 1 \otimes X_i$$
$$\Delta_k : \mathfrak{F} \to \mathfrak{F}^{\otimes^g k}$$
$$\Delta_k = (\Delta \otimes 1) \circ \Delta_{k-1}$$
$$\Delta_k X_i = X_i \otimes 1 \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes 1 \cdots \otimes 1 \otimes X_i$$

If $A : [1, k] \to I$ and $X_A = X_{A(1)} \cdots X_{A(k)}$ then

$$\Delta_p X_A = \sum_{S_1 + \cdots + S_p = [1,k]} \varepsilon(S_1, \cdots, S_p) X_{A|S_1} \otimes \cdots \otimes X_{A|S_p}$$

$S_1 + \cdots + S_p = [1, k]$:The S_i are disjoint and their union is $[1, k]$.
$\varepsilon(S_1, \cdots, S_p) = \pm 1$ is the sign of the permutation

$$\begin{pmatrix} 1, & \cdots & , k \\ S_1, & \cdots & , S_p \end{pmatrix}$$

If $\mathfrak{B}$ is an algebra and $M : \mathfrak{B} \otimes \mathfrak{B} \to \mathfrak{B}$ is the multiplication, define the convolution in the space $\mathcal{L}(\mathfrak{F}, \mathfrak{B})$ of linear mappings $\mathfrak{F} \to \mathfrak{B}$

$$\varphi * \psi = M \circ (\varphi \otimes \psi) \circ \Delta$$

With convolution $\mathcal{L}(\mathfrak{F}, \mathfrak{B})$ is an associative algebra.
The unit is $1_{\mathfrak{B}}\delta$.

$$\varphi_1 * \cdots * \varphi_k = M_k \circ (\varphi_1 \otimes \cdots \otimes \varphi_k) \circ \Delta_k$$

Limit theorems

We consider a 2-graded algebra $\mathfrak{A}$and $a_1, \cdots, a_n \in \mathfrak{A}^1$ and a 2-graded algebra $\mathfrak{B}$ and an *even* functional $\omega : \mathfrak{A} \to \mathfrak{B}$ and with $\omega(1) = 1$ and for a fixed s

$$\omega(a_{i_1}) = 0, \cdots, \omega(a_{i_1} \cdots a_{i_{s-1}}) = 0$$

We want to calculate for $N \to \infty$ and $f \in \mathfrak{F}, f = X_A = X_{A(1)} \cdots X_{A(k)}$ the limit of

$$F_N(f) = M_N \circ \omega^{\otimes N}(f(X_i \mapsto N^{-1/s}(a_i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes a_i))$$

Set

$$C_N(f) = M_N \circ \omega^{\otimes N}(f(X_i \mapsto a_i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes a_i)) \in \mathfrak{B}.$$

Define the homomorphism $\eta : \mathfrak{F} \to \mathfrak{A} : X_i \mapsto a_i$. Then

$$C_N(f) = M_N \circ \omega^{\otimes N} \circ \eta^{\otimes N} \circ \Delta_N(f) = (\omega \circ \eta)^{*N}(f).$$

Set $1_{\mathfrak{B}}\delta = \delta$

$$\psi = \omega \circ \eta - 1_{\mathfrak{B}}\delta = \psi - \delta.$$

$$\psi(1) = 0, \psi(a_{i_1}) = 0, \cdots, \psi(a_{i_1} \cdots a_{i_{s-1}}) = 0$$

Then

$$(\omega \circ \eta)^{*N} = (\delta + \psi)^{*N} = \delta + \sum_{p=1}^{N} \binom{N}{p} \psi^{*p} = \delta + \sum_{p=1}^{N} \binom{N}{p} M_p \circ (\psi^{\otimes p}) \circ \Delta_p$$

For $A : [1, k] \to I$

$$M_p \circ (\psi^{\otimes p} \circ \Delta_p)(X_A) = \sum_{S_1 + \cdots S_p = [1,k]; \# S_i \geq s} \varepsilon(S_1, \cdots, S_p) \psi(X_{A|S_1}) \otimes \cdots \otimes \psi(X_{A|S_p})$$

and

$$F_N(X_A) = N^{-k/s} C_N(X_A) = \sum_{p=1}^{N} \binom{N}{p} N^{-k/s}$$
$$\sum_{S_1+\cdots S_p=[1,k];\#S_i\geq s} \varepsilon(S_1,\cdots,S_p)\psi(X_{A|S_1})\otimes\cdots\otimes\psi(X_{A|S_p})$$

So $F_N(X_A)$ vanishes unless $ps \leq k$ or $p \leq k/s$. For $p \leq k/s$ and $N \to \infty$ we have

$$N^{-k/s}\binom{N}{p} \to \begin{cases} 1/p! & \text{for } p = k/s \\ 0 & \text{for } p < k/s \end{cases}$$

So $F_N(X_A) \to 0$, if k is not a multiple of s. For $k = ps$

$$F_N(X_A) \to \frac{1}{p!} M_p \circ \psi^{\otimes p}$$
$$\Big(\sum_{\substack{S_1 + \cdots S_p = [1,k]; \\ \# S_i = s}} \varepsilon(S_1, \cdots, S_p) X_{A|S_1} \otimes \cdots \otimes X_{A|S_p} \Big).$$

Consider $\mathfrak{F}_s$, the set of homogeneous polynomials of degree s and the algebra $\mathfrak{T}(\mathfrak{F}_s)$ spanned by the tensor products of the elements of $\mathfrak{F}_s$. Define

$$\mathfrak{L}_s = \begin{cases} \mathbb{C}\Big[\Xi_{i_1,\cdots,i_s} : i_1, \cdots, i_s \in I\Big] & \text{for } s \text{ even} \\ \bigwedge \Big(\Xi_{i_1,\cdots,i_s} : i_1, \cdots, i_s \in I\Big) & \text{for } s \text{ odd} \end{cases}$$

Put

$$\kappa : \mathfrak{T}(\mathfrak{F}_s) \to \mathfrak{L}_s : X_{i_1} \cdots X_{i_s} \mapsto \Xi_{i_1,\cdots,i_s}$$

If s is even(odd) and T is a symmetric (antisymmetric) tensor in $\mathfrak{T}(\mathfrak{F}_s)$, then there exists a mapping $\iota : \mathfrak{L}_s \to \mathfrak{T}(\mathfrak{F}_s)$ such that $T = \iota \circ \kappa(T)$. The tensor

$$T = X_{A|S_1} \otimes \cdots \otimes X_{A|S_p}$$

is symmetric for s even and antisymmetric for s odd. Hence

$$X_{A|S_1} \otimes \cdots \otimes X_{A|S_p} = \iota(\Xi_{A|S_1} \cdots \Xi_{A|S_p})$$

Define

$$\begin{aligned}
\Gamma_s &: \mathfrak{F} \to \mathfrak{L}_s \\
\Gamma_s(1) &= 1 \\
\Gamma_s(X_A) &= 0 \text{for } k \text{ not a multiple of } s \\
\Gamma_s(X_A) &= 1/p! \sum_{\substack{S_1+\cdots+S_p=[1,k];\\ \#S_i=s}} \varepsilon(S_1, \cdots, S_p)\Xi_{A|S_1} \cdots \Xi_{A|S_p} \quad \text{for } k = ps
\end{aligned}$$

with $X_A = X_{A(1)} \cdots X_{A(k)}$. Define

$$\Theta_s : \mathfrak{F} \to \mathfrak{L}_s$$
$$\Theta_s(X_w) = \begin{cases} \Xi_w & \text{for } \#w = s \\ 0 & \text{for } \#w \neq s \end{cases}$$

Then

$$\Gamma_s = \exp_* \Theta_s.$$

If $\mathfrak{F}' \subset \mathfrak{F}$ is the subspace generated by the monomials $\neq 1$ and $\mathfrak{T}(\mathfrak{F}')$ is the tensoralgebra and

$$\mathfrak{T}(\omega \circ \eta) : \mathfrak{T}(\mathfrak{F}') \to \mathfrak{B}$$

is the corresponding mapping, then

Theorem (Central Limit Theorem).

$$F_N(f) \to \mathfrak{T}(\omega \circ \eta)\iota(\Gamma_s(f))$$

Structure of Γ_s

Define by $\mathfrak{F}\mathfrak{L}$ the subspace of $\mathfrak{F}$ spanned by

$$X_i, [X_i, X_j]^g, [X_i, [X_j, X_k]^g]^g, [X_i, [X_j, [X_k, X_l]^g]^g]^g, \cdots .$$

Prove by induction

Lemma. *For* $f \in \mathfrak{F}\mathfrak{L}$

$$\Delta_p f = f \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes f.$$

Theorem. *Denote by* m *the multiplication* $m : \mathfrak{L}_s \otimes^g \mathfrak{L}_s \to \mathfrak{L}_{\mathfrak{s}}$. *The functional*

$$m \circ (1_{\mathfrak{L}_s} \otimes \Gamma_s) : \mathfrak{L}_s \otimes^g \mathfrak{F} \to \mathfrak{L}_s$$

vanishes on the two-sided ideal generated by elements of the form

$$1 \otimes f - \Theta_s(f) \otimes 1$$

where f runs through all homogeneous polynomials of degree s in $\mathfrak{F}\mathfrak{L}^g$.

If s is even, then the algebra $\mathfrak{L}_s$ is commutative. Identify in the algebra $\tilde{\mathfrak{F}}_s = \mathfrak{L}_s \otimes^g \mathfrak{F}$ the elements $\xi \otimes 1$ with $\xi \in \mathfrak{L}_s$ and the elements $X_i$with $1 \otimes X_i \in \mathfrak{F}$, then

$$X_{i_1} \cdots X_{i_j} \xi X_{i_{j+1}} \cdots X_{i_k} = \xi X_{i_1} \cdots X_{i_j} X_{i_{j+1}} \cdots X_{i_k}.$$

So $\tilde{\mathfrak{F}}_s$ can be considered as the free algebra with $\mathfrak{L}_s$ as coefficient algebra.

If s is odd, ,then $\mathfrak{L}_s$ is anticommutative. We have mutatis mutandis the same situation, except that for homogeneous ξ

$$X_{i_1} \cdots X_{i_j} \xi X_{i_{j+1}} \cdots X_{i_k} = (-1)^{j \operatorname{deg} \xi} \xi X_{i_1} \cdots X_{i_j} X_{i_{j+1}} \cdots X_{i_k}.$$

Identifying $m \circ (1 \otimes \Gamma_s)$ with Γ_s, we may formulate the last theorem as follows

Theorem. *The functional*

$$\Gamma_s : \tilde{\mathfrak{F}}_s \to \mathfrak{L}_s$$

vanishes on the two-sided ideal $\mathfrak{I}$generated by the elements of the form

$$f - \Theta_s(f)$$

where f runs through all homogeneous polynomials of degree s in $\mathfrak{F}\mathfrak{L}^g$.

For $s = 1$ we obtain, that $\mathfrak{L}_1 = \bigwedge \Xi_i; i = 1, \cdots n)$ and Γ_1 is the homorphism

$$X_i \in \tilde{\mathfrak{F}}_1 \mapsto \Xi_i \in \mathfrak{L}_1.$$

For $s = 2$, the ideal $\mathfrak{I}$ is generated by the elements

$$(X_iX_j + X_jX_i) - (\Xi_{ij} + \Xi_{ji}).$$

The algebra $\tilde{\mathfrak{F}}/\mathfrak{I}$ is the algebra generated by $x_i, i = 1, \cdots, n$ with defining relations

$$x_ix_j + x_jx_i = \Xi_{ij} + \Xi_{ji}.$$

So this is a *Clifford-algebra*.

For $s = 3$ the ideal is generated by

$$X_iX_jX_k - X_jX_kX_i + X_iX_kX_j - X_kX_jX_i - (\Xi_{ijk} - \Xi_{jki} + \Xi_{ikj} - \Xi_{kji})$$

Literature v.W. LN Math 1210 1986