The Steiner Ratio
A Report
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Abstract

Steiner’s Problem is the "Problem of shortest connectivity", that means, given a finite set of points in a metric space $X$, search for a network interconnecting these points with minimal length. This shortest network must be a tree and is called a Steiner Minimal Tree (SMT). It may contain vertices different from the points which are to be connected. Such points are called Steiner points.

If we do not allow Steiner points, that means, we only connect certain pairs of the given points, we get a tree which is called a Minimum Spanning Tree (MST).

Steiner’s Problem is very hard as well in combinatorial as in computational sense, but, on the other hand, the determination of an MST is simple. Consequently, we are interested in the greatest lower bound for the ratio between the lengths of these trees:

$$m(X) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \subseteq X \text{ is a finite set} \right\},$$

which is called the Steiner ratio (of the space $X$).

We look for estimates and exact values for the Steiner ratio in several metric spaces.
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Chapter 1

Steiner’s Problem

The problem of "Shortest Connectivity" has a long and convoluted history. In 1836 Gauß [58] asked in a letter to his friend Schuhmacher

Ist bei einem 4Eck ... von dem kürzesten Verbindungssystem die Rede ..., bildet sich so eine recht interessante mathematische Aufgabe, die mir nicht fremd ist, vielmehr habe ich bei Gelegenheit eine Eisenbahnverbindung zwischen Harburg, Bremen, Hannover, Braunschweig...in Erwägung genommen ....

In English: "How can a railway network of minimal length which connects the four German cities Bremen, Harburg (today part of the city of Hamburg), Hannover, and Braunschweig be created?" \(^1\)

The problem seems disarmingly simple, but it is rich with possibilities and difficulties, even in the simplest case, the Euclidean plane. This is one of the reasons that an enormous volume of literature has been published, starting in the seventeenth century and continuing today.

Over the years Steiner’s Problem has taken on an increasingly important role. More and more real-life problems are given which use Steiner’s Problem or one of its relatives as an application, as a subproblem or as a model, compare [32].

1.1 Steiner Minimal Trees

Starting with the famous book "What is Mathematics" by Courant and Robbins the following problem has been popularized under the name of Steiner:

For a given finite set of points in a metric space find a network which connects all points of the set with minimal length.

\(^1\)A picture of this letter can be found on the cover of the book *Approximation Algorithms* [112].
Such a network must be a tree, which is called a Steiner Minimal Tree (SMT). It may contain vertices other than the points which are to be connected. Such points are called Steiner points. If we don’t allow Steiner points, that is if we connect certain pairs of given points only, then we refer to a Minimum Spanning Tree (MST).

The history of Steiner’s Problem started with P.Fermat [51] early in the 17th century and C.F.Gauß [58] in 1836. At first perhaps with the famous book What is Mathematics by R.Courant and H.Robbins in 1941, this problem became popularized under the name of Steiner. A classical survey of Steiner’s Problem in the Euclidean plane was presented by Gilbert and Pollak in 1968 [59] and christened “Steiner Minimal Tree” for the shortest interconnecting network and “Steiner points” for the additional vertices.

Given a set of points, it is a priori unclear how many Steiner points one has to add in order to construct an SMT. Without loss of generality, the following is true for any SMT for a finite set $N$ of points in the Euclidean plane:

1. The degree of each vertex is at most three;
2. The degree of each Steiner point equals three; and two edges which are incident to a Steiner point meet at an angle of 120°;
3. There are at most $|N| - 2$ Steiner points, where equality holds if and only if every given vertex is of degree one;
4. An SMT has at most $2|N| - 3$ edges;
5. An SMT is an MST for the set $N \cup Q$, where $Q$ is the set of Steiner points;
6. It is only necessary to search the Steiner points in the set

$$Q = \{w : \rho(v, w) \leq \text{length (MST for } N\})\},$$

where $v$ is a point of $N$.

It is well-known that solutions of Steiner’s problem depend essentially on the way in which the distances in space are determined. In recent years it turned out that in engineering design it is interesting to consider Steiner’s Problem and similar problems in several two-dimensional Banach spaces and some specific higher-dimensional cases. Moreover, Steiner’s Problem is of interest in several other metric spaces, for instance in graphs [96] and in phylogenetic spaces [32].

Here, a metric space $(X, \rho)$ is characterized by a set $X$ of points equipped by a function $\rho : X \times X \to \mathbb{R}$ satisfying:

(i) $\rho(x, y) \geq 0$ for all $x, y$ in $X$;

(ii) $\rho(x, y) = 0$ if and only if $x = y$;

(iii) $\rho(x, y) = \rho(y, x)$ for all $x, y$ in $X$; and
(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z$ in $X$ (triangle inequality).

Usually, such a function $\rho$ is called a metric. We will say that the quantity $\rho(x, y)$ is the distance between the points $x$ and $y$.

Now, **Steiner’s Problem of Minimal Trees** is the following:

**Given:** A finite set $N$ of points in the metric space $(X, \rho)$.

**Find:** A connected graph $G = (V, E)$ embedded in the space such that

- $N \subseteq V$ and
- the quantity
  \[
  L(G) = L(X, \rho)(G) = \sum_{v \neq v' \in E} \rho(v, v')
  \]  
  (1.1)

  is minimal as possible.

In the last four decades the investigations and, naturally, the publications about Steiner’s Problem have increased rapidly. In this sense, surveys about Steiner’s Problem, in form of monographs, are given by

5. A.O.Ivanov, A.A.Tuzhilin: "Branching Solutions to One-Dimensional Variational Problems", 2000, [75].
8. A.O.Ivanov, A.A.Tuzhilin: "Theory of Extreme Networks", 2003, [74]

There are several collections about Steiner’s Problem and its relatives: [13], [46], [73], [93] and [112]. A nice representation of the complete subject has been given in [9], [33], [63] and [108].

\(^2\)Note that the axioms are not independent: (i) is a consequence of (iv). On the other hand, A metric $\rho$ can be defined equivalently by

(ii) $\rho(x, y) = 0$ if and only if $x = y$; and

(iv') $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ for all $x, y, z$ in $X$. 

5
1.2 Minimum Spanning Trees

If we don’t allow Steiner points, that is if we connect certain pairs of given points only, then we refer to a Minimum Spanning Tree (MST). Starting with Boruvka in 1926 and Kruskal in 1956, Minimum Spanning Trees have a well-documented history [60] and effective constructions [17].

A minimum spanning tree in a graph $G = (N, E)$ with a positive length-function $f : E \rightarrow \mathbb{R}$, can be found with the help of Kruskal’s [78] well-known method:

1. Start with the forest $T = (N, \emptyset)$;
2. Sequentially choose the shortest edge that does not form a circle with already chosen edges;
3. Stop when all vertices are connected, that is when $|N| - 1$ edges have been chosen.

Then an MST for a finite set $N$ of points in a metric space $(X, \rho)$ can be found obtaining the graph $G = (N, (N^2))$ with the length-function

$$f(vw') = \rho(v, w').$$

Consequently, it is easy to find an MST for $N$; this is valid in the sense of the combinatorial structure as well as in the sense of computational complexity. We can find an MST for $n$ points in $O(n^2 \log n)$-time.

There are several minimum spanning tree algorithms for graphs that are asymptotically faster than Kruskal’s algorithms. A complete discussion of minimum spanning tree strategies in networks is given by Tarjan [109], [110]. A survey about MST’s is given by Wu and Chao [118].

1.3 Properties of SMT’s

The following properties are important for the considerations of a Steiner Minimal Trees $T = (V, E)$ for a finite set $N$:

**Observation 1.3.1** The degree of each vertex is at least one.

**Observation 1.3.2** The degree of each Steiner point is at least three.

*Proof.* It is impossible for a Steiner point $v$ to have degree one, since the edge $vv'$ which joins $v$ with the remaining tree has a positive length, contradicts the minimality requirement.

The triangle inequality of the metric $\rho$ implies the assertion in the following way: Let $v$ be a Steiner point of degree two. Then we may replace the two edges $vw$ and $vw'$ by the edge $ww'$. Because

$$\rho(w, w') \leq \rho(w, v) + \rho(v, w'),$$

(1.3)
the new tree is not longer than the old.

\[ \square \]

**Observation 1.3.3** It is sufficient to consider only finite trees as candidates for an SMT.

The proof uses only the both observations above, see [32].

\[ \square \]

**Observation 1.3.4** There are at most \(|N| - 2\) Steiner points, where equality holds if and only if every given vertex is of degree one, and each Steiner point is of degree three.

*Proof.* The assertion is a consequence of

\[
2 \cdot |N| + 2 \cdot |V \setminus N| - 2 \quad = \quad 2 \cdot (|V| - 1)
\]

\[
= \quad 2 \cdot |E|
\]

\[
= \quad \sum_{v \in V} g_T(v)
\]

\[
= \quad \sum_{v \in V \setminus N} g_T(v) + \sum_{v \in N} g_T(v)
\]

\[
\geq \quad 3 \cdot |V \setminus N| + |N|.
\]

The discussion of equality follows immediately.

\[ \square \]

**Observation 1.3.5** The tree has at most \(2|N| - 2\) vertices and \(2|N| - 3\) edges.

Now, we will discuss the relation between the length of an SMT and an MST for a finite set of points. By definition:

**Observation 1.3.6** An MST cannot be longer than an SMT:

\[
L(\text{SMT for } N) \leq L(\text{MST for } N).
\]  

(1.4)

On the other hand,

**Observation 1.3.7** An SMT is an MST for the set \(N \cup Q\), where \(Q\) is the set of Steiner points:

\[
L(\text{SMT for } N) = \inf \{L(\text{MST for } \tilde{N}) : N \subseteq \tilde{N}\}.
\]  

(1.5)

*Proof.* If the Steiner points have been localized, an SMT for \(N\) is simple to find as the MST for all points.
Observation 1.3.8 It is only necessary to search the Steiner points in the set

\[ Q = \{ w : \rho(v,w) \leq L(MST \text{ for } N) \}, \]

where \( v \) is a point of \( N \).

Comparing all these facts, the search for an SMT for a finite set of points in a metric space forces investigations of two specific questions:

- How many Steiner points are used in an SMT?
- Where are these Steiner points located in the space?

Unfortunately, these questions cannot be solved independently from the construction of the shortest tree itself.

Methods to find an SMT for \( N \) are still unknown or at least hard in the sense of computational complexity. In particular for specific finite-dimensional spaces:

<table>
<thead>
<tr>
<th>space</th>
<th>complexity</th>
<th>source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean plane</td>
<td>( NP )-hard</td>
<td>[56]</td>
</tr>
<tr>
<td>Rectilinear plane ( \mathcal{L}_2^2 )</td>
<td>( NP )-hard</td>
<td>[57]</td>
</tr>
<tr>
<td>( \mathcal{L}_p )-planes</td>
<td>algorithm needs exponential time</td>
<td>[34]</td>
</tr>
<tr>
<td>Banach plane</td>
<td>algorithm needs exponential time</td>
<td>[18]</td>
</tr>
</tbody>
</table>

For higher-dimensional spaces the problems are not easier than in the planes. For a complete discussion of these difficulties see [21] and [68]. Moreover, to solve Steiner’s Problem we need facts about the geometry of the space. On the other hand, for an MST we only use the mutual distances between the points.
Chapter 2

The Steiner Ratio

2.1 The interest in the ratio

Over the years Steiner’s Problem has taken on an increasingly important role, it is one of the most famous combinatorial-geometrical problems. However, all investigations showed the great complexity of the problem, as well in the sense of structural as in the sense of computational complexity. In other terms, considering Steiner’s Problem in metric spaces:

Observation I.

In general, methods to find an SMT are hard in the sense of computational complexity or still unknown.\footnote{Only in several specific metric spaces Steiner’s Problem is simple.} In any case we need a subtle description of the geometry of the space.

On the other hand, a Minimum Spanning Tree (MST) can be found easily by simple and general applicable methods.

Observation II.

It is easy to find an MST by an algorithm which is simple to realize and running fast in all metric space. The algorithm does not need any geometry of the space, it only uses the mutual distances between the points.

Hence, it is of interest to know what the error is if we construct an MST instead of an SMT. In this sense, we define the Steiner ratio for a metric space $X$ to be the infimum over all finite sets of points of the length of an SMT divided by the length of an MST:

$$m(X) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \text{ a finite set in the space } X \right\}.$$ 

This quantity is a parameter of the considered space and describes the performance ratio of the the approximation for Steiner’s Problem by a Minimum Spanning Tree.
Roughly speaking, \( m(X) \) says how much the total length of an MST can be decreased by allowing Steiner points:

\[
L(X)(\text{SMT for } N) \geq m(X) \cdot L(X)(\text{MST for } N). \tag{2.1}
\]

In other terms, the quantity \( m(X) \cdot L(X)(\text{MST for } N) \) would be a convenient lower bound for the length of an SMT for any set \( N \) in the metric space \( (X, \rho) \).

Note, that there are metric spaces in which not any finite set has an SMT. A simple example: Consider three points \( v_1, v_2 \) and \( v_3 \) which form the nodes of an equilateral triangle in the Euclidean plane. An SMT uses one Steiner point \( q \), which is uniquely determined by the condition that the three angles at this point are equal, and consequently equal 120°. Now, remove \( q \) from the plane, and we cannot find an SMT for \( v_1, v_2 \) and \( v_3 \) in this new metric space.

Then we define the Steiner ratio more carefully:

\[
m(X) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \text{ a finite set in } X \text{ for which an SMT exists} \right\}.
\]

Another point of view: The Steiner ratio is a measure of the geometry of the space related to its combinatoric properties.

### 2.2 The Steiner ratio of metric spaces

It is obvious that \( 0 < m(X, \rho) \leq 1 \) for the Steiner ratio of each metric space \( (X, \rho) \). Of course, for the real line the MST and the SMT are identical, and its Steiner ratio equals 1.

On the other hand, the lower bound can be given sharper:

**Theorem 2.2.1** *(E.F.Moore in [59])*: For the Steiner ratio of every metric space

\[
m(X, \rho) \geq \frac{1}{2}
\]

holds.

**Proof.** Let \( T \) be an SMT for a finite set \( N \). Consider the graph \( G \) obtained by replacing each edge of \( T \) by two parallel edges. Since an even number of edges is

\[\text{We define the Steiner ratio as a relative approximation. An absolute one is senseless, since:}\]

**Observation 2.1.1** *(Widmayer [116])*: Unless \( P = NP \), no polynomial time approximation algorithm \( M \) for Steiner’s Problem in networks can guarantee

\[
L(M(N)) - L(\text{SMT for } N) \leq K, \tag{2.2}
\]

where \( N \) is a given set of vertices in the network, and \( K \) is some fixed constant.

This is, of course, true when we use an MST as approximation for Steiner’s Problem.
incident with each vertex of $G$ the graph $G$ has a Eulerian cycle, which has the length $2 \cdot L(T)$ and is a tour through $N$. This tour is not shorter than a minimal tour in which no Steiner point exists. If we delete any edge of the minimal tour we obtain a tree interconnecting $N$ without Steiner points. Hence,

$$L(\text{MST for } N) \leq 2 \cdot L(T) = 2 \cdot L(\text{SMT for } N) \quad (2.3)$$

which implies the assertion.

Note that the proof of 2.2.1 can be used to show a slightly stronger result, namely

**Corollary 2.2.2** Let $N$ be a finite set of $n$ points in a metric space $(X, \rho)$. Then

$$L(\text{MST for } N) \leq 2 \cdot \left(1 - \frac{1}{n}\right) \cdot L(\text{SMT for } N). \quad (2.4)$$

We will see that the lower bound 0.5 is the best possible one over the class of all metric spaces. But this is not true for specific spaces.

Showing that the Steiner ratio of metric space is less than $3/4$ needs more than three points. Defining

$$m^n(X, \rho) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \subseteq X, |N| \leq n \right\}. \quad (2.5)$$

Then, obviously, this quantity is monotonically decreasing in the value $n$:

$$m^{n+1}(X, \rho) \leq m^n(X, \rho) \quad (2.6)$$

for $n > 2$; and

$$m(X, \rho) = \inf \{m^n(X, \rho) : n \text{ a positive integer}\} \quad (2.7)$$

$$= \lim_{n \to \infty} m^n(X, \rho). \quad (2.8)$$

**Theorem 2.2.3** For any metric space $(X, \rho)$ it holds that

$$m^3(X, \rho) \geq \frac{3}{4}.$$

**Proof.** Let an SMT for a finite set of vertices be given. If there is a Steiner point used then we have a subset $N = \{v_1, v_2, v_3\}$ which creates a star consisting of three edges from $v_1, v_2$ and $v_3$ to the common Steiner point $v$.

Say that $\rho(v_2, v_3)$ is greater than both $\rho(v_1, v_2)$ and $\rho(v_1, v_3)$. Then

$$L_M := L(\text{MST for } N) = \rho(v_1, v_2) + \rho(v_1, v_3).$$
The SMT for $N$ has a length $L_S$ less than $L_M$. Then
\[
4 \cdot L_S = 4 \cdot (\rho(v_1, v) + \rho(v_2, v) + \rho(v_3, v)) \\
= 2 \cdot (\rho(v_1, v) + \rho(v_2, v)) + 2 \cdot (\rho(v_2, v) + \rho(v, v_3)) \\
+ 2 \cdot (\rho(v_3, v) + \rho(v, v_1)) \\
\geq 2 \cdot (\rho(v_1, v_2) + \rho(v_2, v_3) + \rho(v_3, v_1)) \\
\geq 2L_M + 2\rho(v_2, v_3) \\
\geq 2L_M + \rho(v_1, v_2) + \rho(v_1, v_3) \\
\geq 3L_M.
\]

Adding the other parts of the tree don’t decrease the ratio.

\[\Box\]

**Conjecture 2.2.4** For any metric space $(X, \rho)$ it holds that
\[
m^n(X, \rho) \geq \frac{n}{2(n-1)}.
\]

This conjecture is true in normed planes: Du et.al. [44].

Assuming that 2.2.4 is true, we have two consequences:
- To show that a metric space has Steiner ratio 2/3, we need a four-point set.
- To show that a metric space has Steiner ratio 1/2, we need a set of arbitrary large cardinality.

## 2.3 The achievement of the Steiner ratio

We said that a (finite) set $N_0$ of points in a metric space $(X, \rho)$ achieves the Steiner ratio if
\[
\frac{L(\text{SMT for } N_0)}{L(\text{MST for } N_0)} = m(X, \rho)
\tag{2.11}
\]

Here, we define for a finite set $N$ of points in $(X, \rho)$
\[
\mu(N) = \mu(N)(X, \rho) = \frac{L(\text{SMT for } N)}{L(\text{MST for } N)}.
\tag{2.12}
\]

Consider the proof: "Inflate" the edges of an SMT $T$ for $N$ to have the width $\epsilon$. Thus, $T$ becomes a polygonal region with a boundary. Suppose that $v_1, \ldots, v_n$ are the given points labeled in counterclockwise order on the boundary. Consider $n$ spanning trees each of which is obtained by deleting an edge from the cycle $v_1, v_2, \ldots, v_n, v_1$. The total length of these $n$ spanning trees is
\[
(n - 1) \cdot \text{length of the cycle.}
\tag{2.9}
\]

Moreover the length of the cycle is less than $2 \cdot L_B(T)$. Therefore, for an MST $T'$ for $N$, we have
\[
2(n - 1) \cdot L_B(T) \geq n \cdot L_B(T').
\tag{2.10}
\]

Is there a hint for the proof of 2.2.4?
Obviously
\[ \mu(N) \geq m(X, \rho) \]  \hspace{1cm} (2.13)
\[ m(X, \rho) = \inf\{\mu(N) : N \subseteq X\} . \]  \hspace{1cm} (2.14)
An immediately consequence of 2.2.2 is

Corollary 2.3.1 Let \((X, \rho)\) be a metric space with Steiner ratio \(1/2\).\(^4\) Then there does not exist a finite set of points in \(X\) which achieves the Steiner ratio.

In other terms, we have to find a sequence \(N_0, N_1, N_2, \ldots\) of finite sets such that

\[ \mu(N_i) \to \frac{1}{2} , \]  \hspace{1cm} (2.15)

to show that the Steiner ratio of a metric space is 0.5.

\(^4\)We will see that such spaces indeed exist.
Chapter 3

The Steiner ratio of Banach-Minkowski-spaces

This present chapter concentrates on investigating the Steiner ratio for Banach spaces. The goal is to determine or at least to estimate the Steiner ratio for many different spaces. We distinguish between finite-dimensional Banach spaces, so-called Banach-Minkowski spaces, and general ones.\(^1\) Our focus on Banach-Minkowski spaces comes from

1. Steiner’s Problem in Banach-Minkowski spaces is of great practical interest, see [13], [24], [27]. Hence, it is good investigated and we have many helpful knowledge about SMT’s.

2. In Banach-Minkowski spaces for any finite set of points an SMT always exist, hence the Steiner ratio is well-defined. In general spaces this must not be true.\(^2\)

3.1 Norms and Balls

Obviously, Steiner’s Problem depends essentially on the way how the distances in the plane are determined. In the present paper, at first, we consider finite-dimensional Banach spaces. These are defined in the following way: \(A_d\) denotes the \(d\)-dimensional affine space with origin \(o\). That means; \(A_d\) is a set of points and these points act over a \(d\)-dimensional linear space. We identify each point with its vector with respect to the origin. In other words, elements of \(A_d\) will be called either points when considerations have a geometrical character, or vectors when algebraic operations are applied. In this sense the zero-element \(o\) of the linear space is the origin of the affine space.

\(^1\)The Steiner ratio of metric spaces lies precisely in the range between 0.5 and 1. We will prove it later. Moreover Ivanov, Tuzhilin, [74] showed that for any real number between 0.5 and 1 there is a metric space with this Steiner ratio. This is not true for Banach spaces.

\(^2\)Compare [6].
two-dimensional affine space is called a plane. A non-empty subset of a affine space which is itself an affine space is called an affine subspace.

The idea of normed spaces is based on the assumption that to each vector of a space can be assigned its "length" or norm, which satisfies some "natural" conditions. A convex and compact body \( B \) of the \( d \)-dimensional affine space \( A_d \) centered in the origin \( o \) is called a unit ball, and induces a norm \( ||\cdot|| = ||\cdot||_B \) in the corresponding linear space by the so-called Minkowski functional:

\[
||v||_B = \inf\{ t > 0 : v \in tB \} \quad \text{for any } v \in A_d \setminus \{o\}, \text{ and}
\]

\[
||o||_B = 0.
\]

On the other hand, let \( ||\cdot|| \) be a norm in \( A_d \), which means:

\[
||\cdot|| : A_d \to \mathbb{R} \text{ is a real-valued function satisfying}
\]

(i) positivity: \( ||v|| \geq 0 \) for any \( v \) in \( A_d \);

(ii) identity: \( ||v|| = 0 \) if and only if \( v = o \);

(iii) homogenity: \( ||tv|| = |t| \cdot ||v|| \) for any \( v \) in \( A_d \) and any real \( t \);

and

(iv) triangle inequality: \( ||v + v'|| \leq ||v|| + ||v'|| \) for any \( v, v' \) in \( A_d \).

Then \( B = \{ v \in A_d : ||v|| \leq 1 \} \) is a unit ball in the above sense. It is not hard to see that the correspondences between unit balls \( B \) and norms \( ||\cdot|| \) are unique. That means that a norm is completely determined by its unit ball and vice versa. Consequently, a Banach-Minkowski space is uniquely defined by an affine space \( A_d \) and a unit ball \( B \). This Banach-Minkowski space is abbreviated as \( M_d(B) \). In each case we also have the induced norm \( ||\cdot||_B \) in the space.

A Banach-Minkowski space \( M_d(B) \) is a complete metric linear space if we define the metric by

\[
\rho(v, v') = ||v - v'||_B. \quad (3.1)
\]

Usually, a (finitely- or infinitely-dimensional) linear space which is complete with regard to its given norm is called a Banach space. Essentially, every Banach-Minkowski space is a finite-dimensional Banach space and vice versa.

All norms in a finite-dimensional affine space induce the same topology, the well-known topology with coordinate-wise convergence.\(^3\) In other words: In such spaces all norms are topologically equivalent, i.e. there are positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \cdot ||\cdot|| \leq ||\cdot||_B \leq c_2 \cdot ||\cdot|| \quad (3.2)
\]

\(^3\)This is the topology derived from the Euclidean metric.
for the two norms \(||.||| \) and \(||.||.||\).
Conversely, there is exactly one topology that generates a finite-dimensional linear space to a metric linear space satisfying the separating property by Hausdorff.

Let \(M_d(B)\) and \(M_d(B')\) be Banach-Minkowski spaces. \(M_d(B)\) is said to be isometric to \(M_d(B')\) if there is a mapping \(\Phi : A_d \to A_d\) (called an isometry) which preserves the distances:

\[
||\Phi v - \Phi v'||_{B'} = ||v - v'||_B \tag{3.3}
\]

for all \(v, v'\) in \(A_d\).
It is easy to see that \(\Phi\) is also an injective mapping. Moreover, a well-known fact given by Mazur and Ulam says that each isometry mapping a Banach-Minkowski space onto another, such that it maps \(o\) on \(o\), is a linear operator. Hence, \(M_d(B)\) is isometric to \(M_d(B')\) if and only if there is an affine map \(\Phi : A_d \to A_d\) with

\[
\Phi B = B'. \tag{3.4}
\]

Consequently,

\[
||\Phi v||_{\Phi B} = ||v||_B. \tag{3.5}
\]

Moreover, the affine map \(\Phi\) is the isometry itself, see Busemann [12].

Steiner’s Problem looks for a shortest network interconnecting a finite set of points, and thus, in particular for a shortest length of a curve \(C\) joining two points. For our purpose, we regard a geodesic curve as any curve of shortest length. If we parametrize the curve \(C\) by a differentiable map \(\gamma : [0,1] \to \mathbb{R}^d\) we define

\[
\text{length of } C = \int_0^1 ||\dot{\gamma}|| dt. \tag{3.6}
\]

It is not hard to see that among all differentiable curves \(C\) from the point \(v\) to the point \(v'\) the segment

\[
\overline{vv'} = \{tv + (1-t)v' : 0 \leq t \leq 1\} \tag{3.7}
\]

minimizes the length of \(C\).

A unit ball \(B\) in an affine space is called strictly convex if one of the following pairwise equivalent properties is fulfilled:

- For any two different points \(v\) and \(v'\) on the boundary of \(B\), each point \(w = tv + (1-t)v', 0 < t < 1\), lies in \(\text{int}B\).

- No segment is a subset of \(\text{bd}B\).

- \(||v + v'||_B = ||v||_B + ||v'||_B\) for two vectors \(v\) and \(v'\) implies that \(v\) and \(v'\) are linearly dependent.

One property more we have in
Lemma 3.1.1 All segments in a Banach-Minkowski space are shortest curves (in the sense of inner geometry). They are the unique shortest curves if and only if the unit ball is strictly convex.

Hence, we can define the metric in a Banach-Minkowski space $M_d(B)$ by

$$\rho(v, v') = \frac{2 \cdot ||v - v'||_{B(2)}}{||w - w'||_{B(2)}},$$

(3.8)

where $ww'$ is the Euclidean diameter of $B$ parallel to the line through $v$ and $v'$ and $||.||_{B(2)}$ denotes the Euclidean norm.

A function $F$ defined on a convex subset of the affine space is called a convex function if for any two points $v$ and $v'$ and each real number $t$ with $0 \leq t \leq 1$, the following is true

$$F(tv + (1-t)v') \leq tF(v) + (1-t)F(v').$$

(3.9)

A function $F$ is called a strictly convex function, if the following is true for any two different points $v$ and $v'$ and each real number $t$ with $0 < t < 1$:

$$F(tv + (1-t)v') < tF(v) + (1-t)F(v').$$

(3.10)

A norm is a convex function. Moreover, the unit ball of a strictly convex norm is a strictly convex set.

Lemma 3.1.2 For a norm $||.||$ in a finite-dimensional affine space the following holds:

(a) A norm $||.||$ in a finite-dimensional affine space is a convex and thus a continuous function.

(b) A norm $||.||$ is a strictly convex function if and only if its unit ball $B = \{v \in A_d : ||v|| \leq 1\}$ is a strictly convex set.

$(.,.)$ denotes the standard inner product, that means for $v = (x_1, \ldots, x_d)$ and $w = (y_1, \ldots, y_d)$ in $A_d$ we define

$$(v, w) = \sum_{i=1}^{d} x_i y_i.$$  (3.11)

Then the Euclidean norm $||.||_{B(2)}$ can be defined by

$$||v||_{B(2)} = \sqrt{(v,v)}.$$  (3.12)

The dual norm $||.||_{DB}$ of the norm $||.||_B$ is defined as

$$||v||_{DB} = \max_{w \neq 0} \frac{(v,w)}{||w||_B}.$$  (3.13)
and has the unit ball $DB$, called the dual unit ball, which can be described as

$$DB = \{ w : (v, w) \leq 1 \text{ for all } v \in B \}. \quad (3.14)$$

Immediately, we have that for any two vectors $v$ and $w$ the inequality

$$(v, w) \leq ||v||_{DB} \cdot ||w||_B; \quad (3.15)$$

is true and it is not hard to see that $B \subseteq B'$ holds if and only if $DB' \subseteq DB$.

An example of non-Euclidean norms dual to each other is

$$|| (t_1, \ldots, t_d) ||_B(\infty) = \max\{|t_1|, \ldots, |t_d|\} \quad (3.16)$$

and

$$|| (t_1, \ldots, t_d) ||_{DB(\infty)} = || (t_1, \ldots, t_d) ||_B(1) = |t_1| + \ldots + |t_d|, \quad (3.17)$$

whereby $B(\infty)$ is a hypercube and $B(1)$ is a cross-polytope.

Particularly, we consider finite-dimensional spaces with $p$-norm, defined in the following way: Let $A_d$ be the $d$-dimensional affine space. For the point $v = (x_1, \ldots, x_d)$ we define the norm by

$$||v||_B(p) = \left( \sum_{i=1}^{d} |x_i|^p \right)^{1/p}$$

where $1 \leq p < \infty$ is a real number. If $p$ runs to infinity we get the so-called Maximum norm

$$||v||_B(\infty) = \max\{|x_i| : 0 \leq i \leq d\}$$

In each case we obtain a Banach-Minkowski space shortly written by $L_d^p$.

$L_1^d$ and $L_{\infty}^d$ normed by a cross-polytope and a cube, respectively. For $1 < p < \infty$ the space $L_p^d$ is strictly convex. The spaces $L_p^d$ and $L_q^d$ with $1/p + 1/q = 1$ are dual, also for the values $p = 1$ and $q = \infty$. The space $L_2^d$ is self-dual.

### 3.2 Steiner’s Problem and SMT’s

A graph $G = (V, E)$ with the set $V$ of vertices and the set $E$ of edges is embedded in a Banach space normed by $||.||$ in the sense that

- $V$ is a finite set of points in the space;
- Each edge $vw' \in E$ is a segment $\{ t v + (1 - t) v' : 0 \leq t \leq 1 \}$, $v, v' \in V$; and
- The length of $G$ is defined by

$$L(G) = \sum_{v,v' \in E} ||v - v'||.$$ 

Now, Steiner’s Problem of Minimal Trees is the following:
**Given:** A finite set $N$ of points in the Banach space.

**Find:** A connected graph $G = (V, E)$ embedded in the space such that
- $N \subseteq V$ and
- $L(G)$ is minimal as possible.

A solution of Steiner’s Problem is called a Steiner Minimal Tree (SMT) for $N$ in the space.

That for any finite set of points there an SMT always exists is not obvious. Particularly, for finite-dimensional spaces it is proved in [21].

The vertices in the set $V \setminus N$ are called Steiner points. We may assume that for any SMT $T = (V, E)$ for $N$ the following holds:

1. The degree of each vertex is at least one;
2. The degree of each Steiner point is at least three; and
3. $|V \setminus N| \leq |N| - 2$. (3.18)

In Banach-Minkowski spaces the condition of length-minimality forces that the degree of the vertices are bounded from above; we quote results about upper bounds of these degrees, depending on the space $M_d(B)$ only. The following table gives some examples of known values for the maximum degree, compare [106].

<table>
<thead>
<tr>
<th>unit ball</th>
<th>degree of a Steiner point</th>
<th>degree of a vertex</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>cube</td>
<td>$2^d$</td>
<td>$2^d$</td>
</tr>
<tr>
<td>cross-polytope</td>
<td>$2d$</td>
<td>$2d$</td>
</tr>
</tbody>
</table>

Let $z(d)$ be the maximum possible degree of a vertex and $s(d)$ be the maximum possible degree of a Steiner point in an SMT in a $d$-dimensional normed space, respectively. Cieslik [19], [21] has shown that $z(d)$ really exists; namely he proved

$$z(d) \leq 3^d - 1,$$ (3.19)

and conjectured

**Conjecture 3.2.1**

$$z(d) \leq 2 \cdot (2^d - 1).$$ (3.20)

It is not hard to see that $2^d \leq s(d) \leq z(d)$, and Morgan [91], [92] conjectured

**Conjecture 3.2.2**

$$s(d) \leq 2^d.$$ (3.21)
Swanepoel [107], in recent times, gives the previously best known upper bound
\[ z(d) \leq O(2^d d^2 \log d). \] (3.22)
Both conjectures (3.20) and (3.21) are true in the planar case, [19], [106], that means:
\[ z(2) = 6 \quad \text{and} \quad s(2) = 4. \]
This gives an approach to reduce Steiner’s Problem in Banach-Minkowski planes to simpler ones.\(^6\)
The two-dimensional methods are very special and offer no hope for generalizations to higher dimensions.\(^5\)

Further investigations for determining these quantities more exactly for specific spaces are necessary, since these numbers have a deep influence in creating fast approximations for shortest networks, compare [28].

### 3.3 The Steiner ratio of specific spaces

We are interested in the value
\[
m(d) = m_d(B) := \inf \left\{ \frac{L_B(\text{SMT for } N)}{L_B(\text{MST for } N)} : N \subseteq M_d(B) \text{ is a finite set} \right\}, \tag{3.24}
\]
which is called the Steiner ratio of the space \( M_d(B) \).\(^7\)

The quantity \( m_d(B) \cdot L(\text{MST for } N) \) would be a convenient lower bound for the length of an SMT for \( N \) in the space \( M_d(B) \); that means, roughly speaking, \( m_d(B) \) says how much the total length of an MST can be decreased by allowing Steiner points.

For the space \( \mathbb{L}_p^d \) the Steiner ratio will be briefly written by \( m(d, p) \).

---

\(^5\)Let \( T = (V, E) \) be a full Steiner tree for the set \( N = \{v_1, \ldots, v_n\}, n > 2, \) of given points. Then let \( V = \{v_1, \ldots, v_{2n-2}\} \), whereby \( g(v_i) = 1 \) for \( i = 1, \ldots, n \) and \( g(v_i) = 3 \) for \( i = n + 1, \ldots, 2n - 2 \). Let \( A(T) = (a_{ij})_{i,j=1,\ldots,2n-2} \) be the adjacency matrix of \( T \). Then it is only necessary to minimize the function
\[
S_B(T) = S_B(v_{n+1}, \ldots, v_{2n-2}) := \sum_{i=1}^{n} \sum_{j=n+1}^{2n-2} a_{ij} ||v_i - v_j||_B + \sum_{i=n+1}^{2n-3} \sum_{j=i+1}^{2n-2} a_{ij} ||v_i - v_j||_B. \tag{3.23}
\]
Compare Cieslik [31].

\(^6\)A similar quantity is maximum possible degree of a vertex in an MST, see [26]. Here \( 3^d - 1 \) is a sharp upper bound, achieved by the hypercube as unit ball, which creates the supremum norm [36].

\(^7\)For infinite-dimensional Banach spaces the Steiner ratio will be defined more carefully later.
I. In the $d$-dimensional affine space $A_d$, the unit ball $B(1)$ is the convex hull of

$$N = \{ \pm (0,\ldots,0,1,0,\ldots,0) : \text{the } i\text{'th component is equal to } 1, \text{ } i = 1,\ldots,d \}. \quad (3.25)$$

The set $N$ contains $2d$ points. The rectilinear distance of any two different points in $N$ equals 2. Hence, an MST for $N$ has the length $2(2d-1)$. Conversely, an SMT\footnote{that this tree is indeed an SMT is not simple to see!} for $N$ with the Steiner point $o = (0,\ldots,0)$ has the length $2d$:

$$\mu(N) \leq \frac{2d}{2(2d-1)} = \frac{d}{2d-1}. \quad (3.26)$$

This implies

**Theorem 3.3.1** For the Steiner ratio of spaces with rectilinear norm the following is true.

$$m(d,1) \leq \frac{d}{2d-1}. \quad (3.27)$$

**Conjecture 3.3.2** (Graham and Hwang [61]) In (3.27) always equality holds.

This is true in the planar case, which means

$$m(2,1) = \frac{2}{3}; \quad (3.28)$$

shown by Hwang [66], but the methods by Hwang do not seem to be applicable to proving the conjecture in the higher dimensional case.

Since $d/(2d-1)$ runs to $1/2$ when $d$ go to infinity, we find together with 2.2.1

**Corollary 3.3.3** The lower bound $1/2$ is the best possible for the Steiner ratio over the class of all Banach-Minkowski spaces.

II. Let $M_d(B)$ and $M_d(B')$ be Banach-Minkowski spaces. A surjective mapping $\Phi : M_d(B) \to M_d(B')$ with the property

$$||\Phi v - \Phi v'||_{B'} = ||v - v'||_B \quad (3.29)$$

for all $v,v'$ in $A_d$ is called an isometry. It is easy to see that $\Phi$ must be injective and

$$\Phi B = B'. \quad (3.30)$$

In other terms, (3.29) and (3.30) are equivalent.

Obviously,

**Lemma 3.3.4** If there exists an isometry between the Banach-Minkowski spaces $M_d(B)$ and $M_d(B')$, then

$$m_d(B) = m_d(B'). \quad (3.31)$$
This relatively simple fact has a lot of interesting consequences:

- Every parallelogram \( B \) in the affine plane \( A_2 \) is the image of the "square" \( B(1) \) under an affine transformation. Consequently, it induces the same Steiner ratio, namely the Steiner ratio of the plane with rectilinear norm and the plane with maximum norm:

\[
m_2(B) = m(L_1^2) = m(L_\infty^2). \tag{3.32}
\]

Whereas in the plane a hypercube and a cross-polytope are "squares", these bodies in higher-dimensional spaces are different, that means, that there does not exist a affine map which transforms one into the other. That is, \( L_1^d \) is not isometric to \( L_\infty^d, \ d \geq 3. \)

- All ellipsoids \( B \) in the affine space \( A_d \) induce the same Steiner ratio, namely the Steiner ratio of the Euclidean space:

\[
m_d(B) = m(L_2^d). \tag{3.33}
\]

- Let \( B \) and \( B' \) be two unit balls in the same affine space \( A_d \). \( B \) and \( B' \) are called similar if \( B = cB' \) for some positive real number \( c \). The lemma implies that the Steiner ratios are equal:

\[
m_d(B) = m_d(B'). \tag{3.34}
\]

III. Let \( M_d(B) \) be a \( d \)-dimensional Banach-Minkowski space, and let \( A_{d'} \) be a \( d' \)-dimensional affine subspace (\( d' \leq d \)) with \( o \in A_{d'} \). Clearly, the intersection \( B \cap A_{d'} \) can be considered as the unit ball of the space \( A_{d'} \). This means that \( M_d(B) \cap A_{d'} \) is a (Banach-Minkowski) subspace of \( M_d(B) \).

Let \( v \) and \( v' \) be two different points in \( A_{d'} \). Then the line through \( v \) and \( v' \) lies completely in \( A_{d'} \), and in view of 3.1.1 and (3.8) we see that the distance between the points \( v \) and \( v' \) is preserved:

\[
\|v - v'\|_B = \|v - v'\|_{B \cap A_{d'}}. \tag{3.35}
\]

Kruskal's method, which finds an MST, uses only the mutual distances between the points. Hence, it holds that

\[
L(B)(\text{MST for } N) = L(B \cap A_{d'})(\text{MST for } N)
\]

for any finite set \( N \) of points in \( M_d(B \cap A_{d'}) \). On the other hand, it is possible that an SMT for \( N \) in the space \( M_d(B) \) is shorter than in the subspace \( M_d(B \cap A_{d'}) \).\(^9\)

Consequently,

\[
L(B)(\text{SMT for } N) \leq L(B \cap A_{d'})(\text{SMT for } N)
\]

for any finite set \( N \) of points in \( M_d(B \cap A_{d'}) \). Then we have

\(^9\)We got an example in the following observation: Let \( N \) be the set of the three points \( v_1 = (1, 0, 0), v_2 = (0, 1, 0) \) and \( v_3 = (0, 0, 1) \) in \( M_3(B(p)) \).

Suppose that the Steiner point of these points lies in the plane determined by \( v_1, v_2 \) and \( v_3 \), that is \( \text{aff}N = \{(x, y, z) : x + y + z = 1 \} \). The strict convexity of the \( p \)-norm has the consequence that there is a unique minimum in this plane; the symmetry of \( v_1, v_2 \) and \( v_3 \) implies that \( v_0 = (1, 0, 3, 3, 1/3) \) is this point. On the other hand, since the function \( F_{N \cdot B(p)}(x, y, z) \) attains its minimum value at
Theorem 3.3.5 Let $M_{d'}(B')$ be a (Banach-Minkowski) subspace of $M_d(B)$. Then

$$m_{d'}(B') \geq m_d(B).$$

3.4 The Banach-Mazur-distance

In (3.2) we said that two norms of a finite-dimensional affine space are equivalent. More exactly: Let $B_d$ denote the class of all unit balls of the $d$-dimensional affine space $A_d$. Since $B$ and $B'$ in $B_d$ are compact bodies, there are positive real numbers $c$ and $c'$ such that

$$\frac{1}{c} \cdot B \subseteq B' \subseteq \frac{1}{c'} \cdot B. \quad (3.36)$$

Hence,

$$c \cdot ||v||_B \geq ||v||_{B'} \geq c' \cdot ||v||_B \quad (3.37)$$

for any $v$ in $A_d$.

Let $N$ be a finite set of points in $A_d$. Assume that $T = (V, E)$ is a tree for $N$, that means $N \subseteq V$. Then

$$c \cdot L(B)(T) = c \cdot \sum_{v' \in E} ||v - v'||_B$$

$$= \sum_{v' \in E} c \cdot ||v - v'||_B$$

$$\geq \sum_{v' \in E} ||v - v'||_{B'}$$

$$= L(B')(T),$$

and similarly, $L(B')(T) \geq c'L(B)(T)$. Consequently, we have

$$c \cdot L_B(T) \geq L_{B'}(T) \geq c' \cdot L_B(T) \quad (3.38)$$

for each tree $T$ for a finite set of points in $A_d$. With these facts in mind, it is easy to see that the following is true:

Theorem 3.4.1 Let $B$ and $B'$ be unit balls in $A_d$ with

$$\frac{1}{c} \cdot B \subseteq B' \subseteq \frac{1}{c'} \cdot B,$$

$v_0$, the following must be true as well:

$$\frac{\partial F_{N,B(p)}(v)}{\partial x}|_{v=v_0} = \frac{\partial F_{N,B(p)}(v)}{\partial y}|_{v=v_0} = \frac{\partial F_{N,B(p)}(v)}{\partial z}|_{v=v_0} = 0,$$

that is

$$-\left(\frac{2}{3}\right)^{p-1} + 2\left(\frac{1}{3}\right)^{p-1} = 0.$$

This implies that $p = 2$. Hence, for $p$ different from 2, the Steiner point does not lie in the plane $\text{aff} N$. 

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where $c, c'$ are positive real numbers. Then
\[ c \cdot m_d(B) \geq m_d(B') \geq \frac{c'}{c} \cdot m_d(B). \]

The Banach-Mazur distance is a natural similarity measure for two Banach spaces. In a first view, we introduce this distance function between classes of Banach-Minkowski spaces in the following way: Let $\mathcal{B}_d$ denote the class of all unit balls in $A_d$, and let $[\mathcal{B}_d]$ be the space of classes of isometries for $\mathcal{B}_d$. Let $j : \mathcal{B}_d \to [\mathcal{B}_d]$ be the canonical mapping. Then the Banach-Mazur distance is a metric on $[\mathcal{B}_d]$ defined as
\[ \Delta([B], [B']) = \ln \inf \{ h \geq 1 : \text{there are } B_1 \in j^{-1}([B]) \text{ and } B_2 \in j^{-1}([B']) \text{ such that } B_1 \subseteq B_2 \subseteq hB_1 \} \]
(3.39)

for $[B], [B']$ in $[\mathcal{B}_d]$.

Let $N$ be a finite set of points in the affine space $A_d$ and let $T = (V, E)$ be a shortest tree for $N$ in $M_d(B)$. Consider the Banach-Minkowski space $M_d(B')$. Suppose that $h = \Delta([B], [B'])$. Then
\[ B \subseteq \Phi(B') \subseteq \exp(h) \cdot B \]
where $\Phi$ is a suitably chosen isometry. With the help of (3.38), we find that
\[ L(B)(T) \geq L(\Phi(B'))(T) \geq \exp(-h)L(B)(T). \]

On the other hand, 3.3.4 says that
\[ L(\Phi(B'))(T) = L(B')(\Phi T), \]
where $\Phi T = (\Phi(V), \Phi(E))$. Consequently,

**Theorem 3.4.2** (Cieslik [27]) Let $B$ and $B'$ be unit balls in the $d$-dimensional affine space $A_d$. Then
\[ e^{\Delta([B], [B'])} \cdot m_d(B) \geq m_d(B') \geq e^{-\Delta([B], [B'])} \cdot m_d(B). \]

**Proof.** There is a sequence $\{h_k\}_{k=1}^{\infty}$ with
\[ h_k \to \exp(\Delta([B], [B'])), \]
where for each number $k$ there are unit balls $B_{1,k} \in j^{-1}([B])$ and $B_{2,k} \in j^{-1}([B'])$ with
\[ B_{1,k} \subseteq B_{2,k} \subseteq h_k B_{1,k}. \]
In view of 3.4.1, this implies the inequalities
\[ h_k \cdot m_d(B_{1,k}) \geq m_d(B_{2,k}) \geq \frac{m_d(B_{1,k})}{h_k}. \]
Together with 3.3.4, we obtain

\[ h_k \cdot m_d(B) \geq m_d(B') \geq \frac{m_d(B)}{h_k}. \]

Hence, if \( k \) tends to infinity, one has the assertion.

\[ \square \]

### 3.5 The Euclidean plane

Consider three points which form the nodes of an equilateral triangle of unit side length in the Euclidean plane. An MST for these points has length 2. An SMT uses one Steiner point. Consequently, with the help of a simple calculation, using the cosine law, we find that the length of the SMT is \( 3 \cdot \sqrt{1/3} = \sqrt{3} \). So we have an upper bound for the Steiner ratio of the Euclidean plane:

\[ \frac{\sqrt{3}}{2} = 0.86602 \ldots \]  

Similarly it is often simple to determine an upper bound for the Steiner ratio of a specific space, since we have only to find a finite set of points with an interconnecting tree shorter than the MST. On the other hand, it will be hard to determine sharp upper bounds, good lower bounds or the exact value of this quantity.

To show this let us consider the history of the determination of the Euclidean Steiner ratio: A long-standing conjecture, given by Gilbert and Pollak in 1968, asserts that in the above inequality (3.41), equality holds; that is \( m = \sqrt{3}/2 \) is the Steiner ratio of the Euclidean plane:

**Conjecture 3.5.1** For the Euclidean plane the following is true:

\[ m_2(B(2)) = \frac{\sqrt{3}}{2} = 0.86602 \ldots \]  

This was the most important conjecture in the area of Steiner’s Problem in the following years. Many people have tried to show this: Pollak [95] and Du, Yao and Hwang [49] have shown that the conjecture is valid for sets \( N \) consisting of \( n = 4 \) points; Du, Hwang and Yao [42] extended this result to the case \( n = 5 \), and Rubinstein and Thomas [97] have done the same for the case \( n = 6 \).

On the other hand, many attempts have been made to estimate the Steiner ratio for the Euclidean plane from below:

\[
\begin{align*}
m &\geq 1/\sqrt{3} & \quad &= 0.57735 \ldots & \quad \text{Graham, Hwang, 1976, [61]} \\
m &\geq \sqrt{2\sqrt{3} + 2 - (7 + 2\sqrt{3})} & \quad &= 0.74309 \ldots & \quad \text{Chung, Hwang, 1978, [16]} \\
m &\geq 4/5 & \quad &= 0.8 & \quad \text{Du, Hwang, 1983, [40]} \\
m &\geq 0.82416 \ldots & \quad &= 0.82416 \ldots & \quad \text{Chung, Graham, 1985, [15]} 
\end{align*}
\]
Finally, in 1990, Du and Hwang [39], [41] created many new methods and said that they succeeded in proving the Gilbert-Pollak conjecture completely.\footnote{This mathematical fact appeared in The New York Times, October 30, 1990 under the title "Solution to Old Puzzle: How Short a Shortcut?"}

But it seems that the proof is not correct. Innami al. [70] describes an mistake. That means, the Gilbert-Pollak-conjecture is still open. But in further considerations we will assume that this conjecture is true.

### 3.6 A bound for $p$-planes

Du and Liu determined an upper bound for the Steiner ratio of $L_p$-planes, using direct calculations of the ratio between the length of SMT’s and the length of MST’s for sets with three elements:

**Theorem 3.6.1** (Du, Liu [84]) The following is true for the Steiner ratio of the $L_p$-planes $M_2(B(p))$:

$$m(2, p) \leq \frac{(2^p - 1)^{1/p} + (2^q - 1)^{1/q}}{4},$$

where $1 < p < \infty$ and $q$ is the conjugate of $p$; that means $\frac{1}{p} + \frac{1}{q} = 1$.

The proof consider the points $u = (1/2, a_p)$, $v = (1, 0)$ and $w = (0, 0)$ with $a_p = (1 - 2^{-p})^{1/p}$. We may assume, that other triangles give better bounds. Now, we will consider another triangle which has a side parallel to the line $\{(x, x) : x \in \mathbb{R}\}$. Let $1 < p < \infty$ and $u = (0, 1)$, $v = (1, 0)$ and $w = (x_p, x_p)$. We wish that the triangle spanned by $u, v$ and $w$ is equilateral and, additionally, $x_p$ lies between 1 and 2. Hence, $x_p$ is a zero of the function $f$ where

$$f(x) = x^p + (x - 1)^p - 2.$$

Of course, $f$ is a strictly monotonically increasing and continuous function. Hence, $f(1) = -1$ and $f(2) = 2^p - 1 > 0$ imply the existence and uniqueness of $x_p$. Then, $L(MST \{u, v, w\}) = 2 \cdot 2^{1/p}$.

\[ \square \]

**Theorem 3.6.2** (Albrecht [1], [3]) Let $1 < p < \infty$, let $x_p$ be a zero of

$$f(x) = x^p + (x - 1)^p - 2,$$

and let $z_p$ minimizing

$$g(z) = 2(z^p + (1 - z)^p)^{1/p} + (x_p - z) \cdot 2^{1/p}.$$  

Then

$$m(2, p) \leq \left(\frac{z_p^p + (1 - z_p)^p}{2}\right)^{1/p} + \frac{1}{2}(x_p - z_p).$$

(3.44)
This result gives the following estimates for $m(2, p)$ for specific values of $p$ and $q$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$3.6.1$</th>
<th>(3.44) with $p$</th>
<th>(3.44) with $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>11</td>
<td>0.782399...</td>
<td>0.775933...</td>
<td>0.775933...</td>
</tr>
<tr>
<td>1.2</td>
<td>6</td>
<td>0.809264...</td>
<td>0.797975...</td>
<td>0.797975...</td>
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<td>0.816708...</td>
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</tr>
<tr>
<td>1.5</td>
<td>2.6</td>
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<td>0.844625...</td>
<td>0.844625...</td>
</tr>
<tr>
<td>1.6</td>
<td>2.4</td>
<td>0.862145...</td>
<td>0.859755...</td>
<td>0.859755...</td>
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<tr>
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</tr>
<tr>
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<td>2.0</td>
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<td>0.866025...</td>
<td>0.866025...</td>
</tr>
<tr>
<td>1.9</td>
<td>1.8</td>
<td>0.865460...</td>
<td>0.865460...</td>
<td>0.865460...</td>
</tr>
</tbody>
</table>

Using only three points, 2.2.3 said that we cannot derive a Steiner ratio less than $3/4$. Hence, we have to investigate sets with four points to get sharper estimates. Albrecht [1] found an upper bound for the Steiner ratio considering the extreme points of the sets $B(1)$ and $B(\infty)$ in $L_2^p$. This idea suggests that we consider the four given points $u = (x_p, 0)$, $v = (0, 1)$, $w = (-x_p, 0)$ and $s = (0, -1)$. Let $q_1 = (a_p, b_p)$ and $q_2 = -q_1$ be Steiner points. The tree $T$ contains the edges $q_1 u$, $q_1 v$, $q_2 w$ and $q_2 s$, since each Steiner point has degree at least three.

**Theorem 3.6.3 (Albrecht [1], [3])** The Steiner ratio of $L_2^p$ is essentially smaller than $3/4$ if $p \leq 1.2$ or if $p \geq 6$.

Albrecht [1] also remarked that neither construction gives an SMT, that means the bounds are upper bounds and never exact values for the Steiner ratio $m(2, p)$.

It is not necessary to do more, that means to use sets of more than four points, since we will see that in general the Steiner ratio of planes is in any case at least $2/3$.

### 3.7 Banach-Minkowski planes

Consider the plane $A_2$ normed by the unit ball $B(1)$. Let $N = \text{ext} B(1)$, that means $N = \{\pm (1, 0), \pm (0, 1)\}$. It is easy to see that

$$
\frac{L_{B(1)}(\text{SMT for } N)}{L_{B(1)}(\text{MST for } N)} \geq \frac{2}{3},
$$

which means that the Steiner ratio of the plane with rectilinear norm may be as small as $2/3$. And moreover, equality holds:

**Theorem 3.7.1 (Hwang [66])** For the plane with rectilinear norm

$$
m_2(B(1)) = \frac{2}{3} = 0.6666 \ldots
$$
holds.

In view of the fact that all parallelograms are affine images of $B(1)$ we have

**Corollary 3.7.2**

$$m_2(B) = \frac{2}{3} = 0.6666\ldots,$$  \hspace{1cm} (3.47)

whenever the unit ball $B$ a parallelogram.

Now we are interested in the best lower bound, than 0.5 for the Steiner ratio of any Banach-Minkowski plane. This bound must be at most $2/3$. Moreover,

**Theorem 3.7.3** (Gao, Du, Graham [55]) For the Steiner ratio of Banach-Minkowski planes the following is true:

$$m_2(B) \geq \frac{2}{3}.$$  
Equality holds if $B$ is a parallelogram.$^{11}$

The proof of the theorem gives a little bit more, since Gao et. al. discuss the equality in 3.7.3.$^{12}$

**Theorem 3.7.4** (Gao, Du, Graham [55]) If there is a natural number $n$ such that the bound $2/3$ is adopted by a set of $n$ points, then $n = 4$, and $B$ is a parallelogram.

In contrast, an upper bound is given by the following theorem.

**Theorem 3.7.5** (Du et.al. [44]) For any unit ball $B$ in the plane the following is true:

$$m_2(B) \leq \frac{\sqrt{13} - 1}{3} = 0.8685\ldots$$  \hspace{1cm} (3.48)

There is no unit ball known which makes the inequality to an equality. And we give

**Conjecture 3.7.6** For any unit ball $B$ in the plane the following is true:

$$m_2(B) \leq \frac{\sqrt{3}}{2} = 0.8665\ldots$$  \hspace{1cm} (3.49)

$^{11}$and only if?

$^{12}$Compare 2.2.4.
3.8 $\lambda$-geometries

It is an interesting question to consider planes which are normed by a regular polygon with an even number of corners. We defined the $\lambda$-geometry $M_2(B^{(\lambda)})$ in the following way: The unit ball $B^{(\lambda)}$ is a regular $2\lambda$-gon, $\lambda > 1$, with the x-axis being a diagonal direction.

Theorem 3.8.1 (Sarrafzadeh, Wong [99]) Assume that 3.5.1 is true. For the Steiner ratio of the planes with $\lambda$-geometry it holds that

$$m_2(B^{(\lambda)}) \geq \frac{\sqrt{3}}{2} \cos \frac{\pi}{2\lambda}.$$  

Proof. Let $N$ be a finite set in $A_2$. Then,

$$L(B^{(\lambda)})(\text{SMT for } N) \geq L(B^{(\infty)})(\text{SMT for } N)$$

$$= L(B(2))(\text{SMT for } N) \text{ using (3.38)}$$

$$\geq \frac{\sqrt{3}}{2} \cdot L(B(2))(\text{MST for } N) \text{ with 3.5.1}$$

$$= \frac{\sqrt{3}}{2} L(B^{(\infty)})(\text{MST for } N)$$

$$\geq \frac{\sqrt{3}}{2} \cos \frac{\pi}{2\lambda} \cdot L(B^{(\lambda)})(\text{MST for } N)$$

Consider $\lambda = 3$. Here, we have first

Lemma 3.8.2 (Laugwitz [79]) Suppose that $B$ is a unit ball in the plane. There is an affinely regular hexagon inscribed in $B$ with vertices on the boundary of $B$.

Proof. The first vertex $p_1$ may be arbitrarily chosen on $\text{bd}B$. We consider the function $\phi : \text{bd}B \to \mathbb{R}$ defined by

$$\phi(v) = ||p_1 - v||_B.$$  

Then $\phi(p_1) = 0$ and $\phi(-p_1) = 2$. Since $\phi$ is a continuous function and $\text{bd}B$ is a compact set there is a point $p_2$ with $\phi(p_2) = 1$. Now it is easy to see that the points $p_1, p_2, p_2 - p_1, -p_1, -p_2$ and $p_1 - p_2$ are the vertices of the desired hexagon.

This gives immediately, see below the proof of 3.10.1:

Theorem 3.8.3

$$m_2(C) \leq \frac{3}{4},$$  

(3.50)

where $C$ is an affinely-regular hexagon.
Since $B^{(3)}$ is an affinely regular hexagon, we obtain

**Corollary 3.8.4** Assume that 3.5.1 is true. Let $B$ be an affinely regular hexagon in the plane. Then

$$m_2(B) = \frac{3}{4}.$$  \hspace{1cm} (3.51)

In view of 3.4.1 we also find

$$m_2(B^{(\lambda)}) \leq \sqrt{\frac{3}{2}} \cdot \frac{1}{\cos \frac{\pi}{2\lambda}}.$$ \hspace{1cm} (3.52)

Thus, paying attention 3.7.5, we have:

**Corollary 3.8.5** Assume that 3.5.1 is true. For the Steiner ratio of the planes with $\lambda$-geometry

$$m_2(B^{(\lambda)}) \leq \min \left\{ \frac{\sqrt{13} - 1}{3}, \frac{\sqrt{3}}{2}, \frac{1}{\cos \frac{\pi}{2\lambda}} \right\}$$

holds.

It is an interesting question to investigate the equality in 3.8.1. Lee and Shen [81] give a complete discussion for the Steiner ratio of the planes with $\lambda$-geometry. Moreover,

**Theorem 3.8.6** (Lee and Shen [81]) Assume that 3.5.1 is true. For the Steiner ratio of the planes with $\lambda$-geometry it holds that

$$m_2(B^{(\lambda)}) = \frac{\sqrt{3}}{2} \cos \frac{\pi}{2\lambda},$$

if $\lambda \equiv 3 \mod 6$, and

$$m_2(B^{(\lambda)}) = \frac{\sqrt{3}}{2},$$

if $\lambda \equiv 0 \mod 6$, $\lambda \geq 6$.

Here we find two phenomenas:

- There are infinitely many different Banach-Minkowski planes which have the same Steiner ratio as the Euclidean plane.

- The Steiner ratio of the planes with $\lambda$-geometry is not a monotonically increasing function of the parameter $\lambda$.

For the following specific (Banach-Minkowski) planes, and only for these, we know the exact value for the Steiner ratio:

\[13\text{in the sense of isometry}\]
3.9 The Steiner ratio of $L_p^3$

In this section we will determine upper bounds for the Steiner ratio of three-dimensional $L_p$-spaces, abbreviated by $m(3, p)$:

$$m(3, p) = m_3(B(p)) = m(L_p^3),$$

(3.53)

$1 \leq p \leq \infty$.

Considering the four points

$$v_1 = (1, 0, 0),$$
$$v_2 = (0, 1, 0),$$
$$v_3 = (0, 0, 1) \text{ and }$$
$$v_4 = (1, 1, 1)$$

which build an equilateral set in the three-dimensional space, we find

**Theorem 3.9.1** (Albrecht [1], [27]) Let $1 < p < \infty$ and let $q$ be the conjugate of $p$. Then we have for the Steiner ratio of $L_p^3$

$$m(3, p) \leq \begin{cases} \frac{1}{3} \left(2^{-1/p} + (2^q - 1)^{1/q}\right) : & 1 < p \leq \frac{\log 3}{\log 3 - \log 2} \\ \left(\frac{2}{3}\right)^{1/q} : & \text{otherwise} \end{cases}$$

On the other hand, using six points

$$v_1 = (x, x - 1, 1 - x),$$
$$v_2 = (x, x, 2 - x),$$
$$v_3 = (1, 0, 1),$$
$$v_4 = (0, 0, 0),$$
$$v_5 = (0, 1, 1) \text{ and }$$
$$v_6 = (x - 1, x, 1 - x),$$

forming a cross-polytope, and adding four Steiner points, we have
Theorem 3.9.2 (Albrecht [1], [27]) Let \( p \) and \( q \) be reals with \( 1 < p < \infty \), \( 1/p + 1/q = 1 \); and let \( x_0 \) be the unique determined zero of the function \( f \) with
\[
f(x) = x^p + 2(x - 1)^p - 2
\]
in the range \((1, 2)\).

Then the Steiner ratio of \( L^3_p \) can be estimated by
\[
m(3,p) \leq \begin{cases} 
\frac{1}{5} \left( (2^q - 1)^{1/q} + \left( \frac{1}{2} \right)^{1/p} + \left( \frac{3}{2} \right)^{1/p} x_0 \right) : & 1 < p \leq \frac{\log 3}{\log 3 - \log 2} \\
\frac{1}{5} \left( \left( \frac{3}{2} \right)^{1/p} (x_0 + 2) \right) : & \frac{\log 3}{\log 3 - \log 2} < p < \infty
\end{cases}
\]

Using theorem 3.9.2 for \( p = \infty \) gives the value \( 3/5 = 0.6 \) for the Steiner ratio, but here we have with help of another consideration the better bound
\[
m(3, \infty) \leq \frac{4}{7} = 0.5714 \ldots
\]

3.10 The range of the Steiner Ratio

An interesting problem, but which seems as very difficult, is to determine the range of the Steiner ratio for \( d \)-dimensional Banach-Minkowski spaces, depending on the value \( d \). More exactly, determine the best possible reals \( c_d \) and \( C_d \) such that
\[
c_d \leq m_d(B) \leq C_d,
\]
for all unit balls \( B \) of \( A_d \), \( d = 1, 2, 3, \ldots \).

Both, the numbers \( C_d \) and \( c_d \), are attained by certain Banach-Minkowski spaces. This follows from the continuity of the Steiner ratio as a function of the space and the Blaschke selection theorem.

The quantity \( C_d \) is defined as the upper bound of all numbers \( m_d(B) \) ranging over all unit balls \( B \) of \( A_d \). Of course, \( C_1 = 1 \), but \( C_2 \) is essentially less since

Theorem 3.10.1 In any Banach-Minkowski space \( M_d(B) \) where \( d \geq 2 \), there is a three point set \( N \) such that the SMT for \( N \) is strictly shorter than an MST for \( N \).

For a proof we start with the observation that it is possible to inscribe a ”regular” hexagon into the unit ball of any Banach-Minkowski plane. Here, ”regular” has two meanings:

1. The hexagon is regular in the sense that all edges have the same length; and
2. It is also affinely regular - an affine image of an Euclidean regular hexagon.

Let \( M_2(B) \) be a Banach-Minkowski plane. In view of 3.8.2 let \( C \) be an inscribed affinely regular hexagon for the unit ball \( B \) such that the nodes \( p_1, \ldots, p_6 \) of \( C \) are placed in this order on the boundary of \( B \). Now we distinguish two cases.
1. $B = C$.

Up to isometry, we may assume that

$$B = \text{conv}\{(1, 1), (-1, -1), (1, 0), (-1, 0), (0, 1), (0, -1)\},$$

(3.55)

which implies that

$$\|(x_1, x_2)\|_B = \max\{|x_1|, |x_2|, |x_1 - x_2|\}.$$

(3.56)

It is easy to see that the set $N = \{p_1, p_3, p_5\}$ has an MST of length 4 and an SMT of length at most 3.

2. Suppose that $C$ is a proper subset of $B$.

Then there is a point $p$ in $\text{bd}B \setminus C$. Without loss of generality we may assume that $p$ lies in the cone spanned by $p_1, o, p_2$. Let $q$ be the only element of the intersection $p_1 \overline{p_2}$ and $op$. Then $\|q\|_B < 1$. Consequently, an SMT for $\{o, p_1, p_2\}$ is strictly shorter than an MST.

Now, we prove the theorem by the following considerations: Let $M_d(B')$ be a (Banach-Minkowski) subspace of $M_d(B)$. The mutual distances between points of $N$ are the same in the spaces $M_d(B \cap A_d')$ and $M_d(B)$. Hence, an MST for $N$ in $M_d(B \cap A_d')$ is an MST in $M_d(B)$ as well. On the other hand, Steiner points in an SMT $T$ for $N$ in $M_d(B)$ can be outside of $A_d'$, such that $T$ is shorter than an SMT for $N$ in $M_d(B \cap A_d')$.

\[ \square \]

**Theorem 3.10.2** A Banach-Minkowski space $M_d(B)$ has Steiner ratio 1 if and only if $d = 1$.

What can we say about higher dimensions?

In a first view it seems that it will be simpler to show the upper rather than the lower bound. In fact this was not the case, it was shown that

- $0.612\ldots \leq c_2 \leq C_2 \leq 0.9036\ldots$ Cieslik, 1990, [20]
- $0.623\ldots \leq c_2 \leq C_2 \leq 0.8686\ldots$ Du, Gao, Graham, Liu, Wan, 1993 [44]
- $0.666\ldots \leq c_2 \leq 0.8686\ldots$ Gao, Du, Graham, 1995 [55]

**Conjecture 3.10.3** For $d = 2, 3, \ldots$

$$C_d = m(d, 2),$$

where $m(d, 2)$ denotes the Steiner ratio of the $d$-dimensional Euclidean space.
This conjecture is open for all values of \( d \), also in the planar case, where we only know
\[
m(2, 2) \leq C_2 \leq \frac{\sqrt{13} - 1}{3},
\]
(3.57)
see 3.7.5, compare [39], [41] and [44].

On the other hand, the quantity \( c_d \) is defined as the lower bound of all numbers \( m_d(B) \) ranging over all unit balls \( B \) of \( A_d \) is of interest. Of course, \( c_1 = 1 \).

**Conjecture 3.10.4** For \( d = 2, 3, \ldots \)
\[
c_d > 1/2.
\]
That means, that there is no Banach-Minkowski space which Steiner ratio achieve the smallest possible value 0.5.

This conjecture is open, except the planar case, where we know
\[
c_2 = \frac{2}{3},
\]
(3.58)
see 3.7.3, compare [55].

Considering the Steiner ratio in \( \mathbb{L}_p^3 \), we find the last conjecture possibly true.

**Theorem 3.10.5** (Albrecht, Cieslik [4]) If the conjectures 3.3.2 and 3.11.3 are true, then the Steiner ratio for each three-dimensional \( \mathbb{L}_p \)-space, \( 1 \leq p \leq 2 \), is essentially greater than 0.5:
\[
m(3, p) > \frac{1}{2}.
\]
(3.59)

### 3.11 The Steiner ratio of Euclidean spaces

In the \( d \)-dimensional Euclidean space, we consider the set \( N \) of \( d + 1 \) nodes of a regular simplex with exclusively edges of unit length. Then an MST for \( N \) has the length \( d \).

It is easy to compute that the sphere that circumscribes \( N \) has the radius
\[
R(N) = \sqrt{d/(2d + 2)}.
\]
(3.60)

With the center of this sphere as Steiner point, we find a tree \( T \) interconnecting \( N \) with the length
\[
L(B(2))(T) = (d + 1)R(N).
\]
(3.61)

Hence, we find by (3.60) and (3.61) the following nontrivial upper bound:
\[
\mu(N) \leq \frac{(d + 1)}{d} \sqrt{\frac{d}{2d + 2}} = \sqrt{\frac{d + 1}{2d}}.
\]
(3.62)

Hence,
Theorem 3.11.1 The Steiner ratio of the \( d \)-dimensional Euclidean space can be bounded as follows:

\[
 m(d, 2) \leq \sqrt{\frac{1}{2} + \frac{1}{2d}}.
\]  

(3.63)

In the proof we used a Steiner point of degree \( d + 1 \), but it is well-known that all Steiner points in an SMT in Euclidean space are of degree 3, compare [21].

A generalized conjecture, posed by Gilbert and Pollak, stated that the Steiner ratio of any Euclidean space was achieved when the given points are the nodes of a regular simplex. The regular simplex is a generalization, to the \( d \)-dimensional Euclidean space, of the two-dimensional triangle and the 3-dimensional tetrahedron. It has \( d + 1 \) nodes and the mutual distances between the nodes of the simplex are equal. In 1992, Smith [103] showed that the generalized Gilbert-Pollak conjecture is false for the dimension \( d \) with \( 3 \leq d \leq 8 \). Moreover, the conjecture is disproved in general by

**Theorem 3.11.2** (Chung, Gilbert [14], Smith [103] and Du, Smith [48]) The Steiner ratio of the \( d \)-dimensional Euclidean space is bounded as follows:

<table>
<thead>
<tr>
<th>dimension</th>
<th>upper bound by Chung, Gilbert</th>
<th>upper bound by Smith</th>
<th>upper bound by Du, Smith</th>
</tr>
</thead>
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<tr>
<td>= 2</td>
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<td></td>
<td></td>
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<td></td>
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<td>= 11</td>
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<td></td>
<td>0.68624...</td>
</tr>
<tr>
<td>= 20</td>
<td>0.69839...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>= 40</td>
<td>0.68499...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>= 80</td>
<td>0.67775...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>= 160</td>
<td>0.67392...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>→ ∞</td>
<td>0.66984...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The first column was computed by Chung and Gilbert considering regular simplices. Here, Du and Smith [48] showed that the regular \( d \)-simplex cannot achieve the Steiner ratio if \( d > 2 \). That means that these bounds cannot be the Steiner ratio of the space when \( d > 2 \).

The second column given by Smith investigates regular octahedra, respectively cross polytopes. Note, that it is not easy to compute an SMT for the nodes of an octahedra. The third column used the ratio of sausages, whereby a sausage is constructed by

1. Start with a ball (of unit diameter) in \( L^{d}_{2} \).
2. Successively add balls so that the \( n \)’th ball you add is always touching the 
\( \min\{d, n - 1\} \) most recently added balls.

This procedure uniquely\(^{14}\) defines an infinite sequence of interior-disjoint numbered 
balls. The centers of these balls form a discrete point set, which is called the (infinity) 
\( d \)-sausage \( N(\infty, d) \). The first \( n \) points of the \( d \)-sausage will be called the ”\( n \)-point 
\( d \)-sausage” \( N(n, d) \). Note, that \( N(d + 1, d) \) is a \( d \)-simplex if \( d \geq 3 \).

Du and Smith [48] present many properties of the \( d \)-sausage, in particular, that

\[
\begin{align*}
    u(d) := & \frac{L(\text{SMT for } N(\infty, d))}{L(\text{MST for } N(\infty, d))} \\
\end{align*}
\]

(3.64)
is a strictly decreasing function of the dimension \( d \).\(^{15}\) Hence, \( u(d), d = 2, 3, \ldots \) is a 
convergent sequence, but the limit is still unknown.

It seems that probably there does not exist a finite set of points in the \( d \)-dimensional 
Euclidean space, \( d \geq 3 \), which achieves the Steiner ratio \( m(d, 2) \). But, if such set in 
spite of it exists, then it must contain exponentially many points. More exactly: Smith 
and McGregor Smith [105] investigate sausages in the three-dimensional Euclidean 
space to determine the Steiner Ratio and following they conjectured that

**Conjecture 3.11.3** For the Steiner Ratio of the three-dimensional Euclidean space

\[
m(3, 2) = \sqrt{\frac{283}{700} - \frac{3\sqrt{21}}{700} + \frac{9\sqrt{11 - \sqrt{21}\sqrt{2}}}{140}}
\]

\[= 0.78419\ldots\]

holds.

These investigations are helpful to discuss the following problem: One of the key 
issues in biochemistry today is predicting the three-dimensional structure of proteins 
from the primary sequence of amino acids. Steiner’s Problem in the three-dimensional 
Euclidean space might help explain the reason for these long molecular chains. In or-
der to examine this potential application area and others related to it, possible linkages 
between the objective function of Steiner’s Problem and objective functions of these 
applications in the biochemical sciences need to be examined, see [88], [89], [90], [105], 
and [111].

\(^{14}\text{up to congruence}\)

\(^{15}\text{Here, we use a generalization of Steiner’s Problem to sets of infinitely many points. This is simple}
\text{to understand. For a finite number of points it is shown that}

\[
\begin{align*}
    \frac{L(\text{SMT for } N(2d + 1, d))}{L(\text{MST for } N(2d + 1, d))} & \leq \frac{L(\text{SMT for } N(d + 1, d))}{L(\text{MST for } N(d + 1, d))},
\end{align*}
\]

which is a finite version of

\[
\begin{align*}
    \frac{L(\text{SMT for } N(\infty, d))}{L(\text{MST for } N(\infty, d))} & \leq \frac{L(\text{SMT for } N(d + 1, d))}{L(\text{MST for } N(d + 1, d))},
\end{align*}
\]

for \( d > 1 \).
Moreover, Du and Smith used the theory of packings to get the following result.\footnote{And gives a partial answer for 2.2.4.}

**Theorem 3.11.4** (Du, Smith [48]) Let $N$ be a finite set of $n$ points in the $d$-dimensional Euclidean space $M_d(B(2))$, $d \geq 3$, which achieves the Steiner ratio $m_d(B(2))$ of the space. Then

$$n \geq \left\lceil \frac{1}{2} \cdot \sqrt{f\left(\frac{\pi}{3}, d\right)} \right\rceil + 1,$$

where

$$f(\theta, d) = \frac{2I_{d-2}(\pi/2)}{I_{d-2}(\theta)}$$

and

$$I_m(x) = \int_0^x (\sin u)^m \, du.$$

3.11.4 implies that the number $n$ grows at least exponentially in the dimension $d$. Some numbers are computed:

<table>
<thead>
<tr>
<th>$d$</th>
<th>$n$ is at least</th>
</tr>
</thead>
<tbody>
<tr>
<td>49</td>
<td>49</td>
</tr>
<tr>
<td>50</td>
<td>53</td>
</tr>
<tr>
<td>100</td>
<td>2218</td>
</tr>
<tr>
<td>200</td>
<td>3481911</td>
</tr>
<tr>
<td>500</td>
<td>$10^{16}$</td>
</tr>
<tr>
<td>1000</td>
<td>$5 \cdot 10^{31}$</td>
</tr>
</tbody>
</table>

When the dimension go to infinity, the Steiner ratio decreases:

**Theorem 3.11.5**

$$1 = m(1, 2) \geq m(2, 2) \geq m(3, 2) \geq m(4, 2) \geq \ldots \geq \lim_{d \to \infty} m(d, 2). \quad (3.65)$$

The sequence $\{m(d, 2)\}_{d=1,2,\ldots}$ is a decreasing, bounded, and consequently, convergent sequence. This immediately implies two questions:

1. When there is in this chain a strict inequality?
2. What is the limit?

A lower bound for the Steiner ratio of Euclidean spaces is given by

**Theorem 3.11.6** (Graham, Hwang [61]) For the Steiner ratio of any Euclidean space

$$m(d, 2) \geq \frac{1}{\sqrt{3}} = 0.57735\ldots$$

holds.
Proof. Let \( N \) be a finite set of points in \( M_d(B(2)) \).

The fact that all Steiner points of an SMT are of degree three implies that it is sufficient only to consider such SMT’s \( T = (V, E) \) which are full trees for \( N \).

Assuming that each vertex in \( Q \) is adjacent to at most one vertex in \( N \). The set \( Q \) induces in \( T \) a subgraph \( G' = (Q, E') \), for which it follows

\[
|E'| = \frac{1}{2} \sum_{v \in Q} g_{G'}(v)
\]

\[
\geq \frac{1}{2} \sum_{v \in Q} (gr(v) - 1)
\]

\[
\geq \frac{1}{2} \sum_{v \in Q} 2
\]

\[
= |Q|.
\]

This contradicts the fact that the forest \( G' \) has at most \( |Q| - 1 \) edges.

In other terms, there is a Steiner point \( q \) in \( T \) with two neighbors \( v, v' \) in \( N \). Without loss of generality, we may assume that \( ||v - q|| \geq ||v' - q|| \). Using the cosine law, it is easily verified that

\[
\frac{||v - q||}{||v' - q'||} \geq \frac{1}{\sqrt{3}}.
\]

Let \( T' \) be an SMT and \( T'' \) an MST for the set \( N \setminus \{v\} \). Then

\[
\frac{L(T)}{L(\text{MST for } N)} \geq \frac{||v - q|| + L(T \text{ without the edge } vq)}{||v' - q'|| + L(T'')}
\]

\[
\geq \frac{||v - q|| + L(T')}{||v' - q'|| + L(T'')}
\]

\[
\geq \min \left\{ \frac{||v - q||}{||v' - q'||}, \frac{L(T')}{L(T'')} \right\}
\]

\[
\geq \frac{1}{\sqrt{3}}
\]

by an induction on the number of points in \( N \).

This lower bound is improved by

**Theorem 3.11.7 (Du [43])** For the Steiner ratio of any Euclidean space

\[
m(d, 2) \geq 0.615 \ldots
\]

holds.

Hence, we are interested in the case when the dimension \( d \) runs to infinity. In the moment we only know by the theorem above and 3.11.1:

\[
\frac{1}{\sqrt{2}} \geq \lim_{d \to \infty} m(d, 2) \geq 0.615 \ldots
\]
3.12 The Steiner ratio of Einstein-Riemann spaces

The so-called Riemannian metric, which is used in differential geometry and in the theory of relativity, is defined with a positive definite matrix \( \Psi = (p_{ij})_{i,j=1,...,d} \) by

\[
||v||_\Psi = (\Psi v, v)^{1/2} = \left( \sum_{i=1}^{d} \sum_{j=1}^{d} p_{ij} x_i x_j \right)^{1/2},
\]

where \( v = (x_1, \ldots, x_d) \).
For \( \Psi = I \) the norm is the Euclidean one.

For positive definite matrices we have

**Lemma 3.12.1** (Horn, Johnson [65]) Let \( \Psi \) be a positive definite matrix and let \( k \geq 1 \) be a given integer. Then there exists a unique positive Hermitian matrix \( \Phi \) such that

\[ \Phi^k = \Psi. \]

Moreover, \( \text{rank } \Phi = \text{rank } \Psi \).

In other terms, each positive definite matrix has a unique \( k \)’th root for all \( k = 1, 2, \ldots \). The most useful case of the preceding lemma is for \( k = 2 \). Here, 3.12.1 can be written in

**Theorem 3.12.2** (Horn, Johnson [65]) Let \( \Psi \) be a positive definite matrix. Then there exists a unique nonsingular matrix \( \Phi \) such that

\[ \Psi = \Phi^* \Phi. \]

The form \( \Phi = \Psi^{1/2} \) is often called the Cholesky decomposition of \( \Psi \).

With these facts in mind, we find

\[
||v||^2_\Phi = (\Phi^* \Phi v, v) = (\Phi v, \Phi^* \Phi v) = (\Phi v, \Phi v) = ||\Phi v||^2_\Phi.
\]

This implies

\[
||v||_\Psi = ||\Phi v||_{\mathbb{R}(2)},
\]

which says, compare (3.5), that \( \Phi \) is an isometry to the Euclidean space. In view of 3.3.4, we have

**Theorem 3.12.3** Let \( M(d, \Phi) \) be a \( d \)-dimensional Einstein-Riemann space normed by positive definite matrix \( \Phi \). Then

\[ m(M(d, \Phi)) = m(d, 2), \]

where \( m(d, 2) \) denotes the Steiner ratio of the \( d \)-dimensional Euclidean space.

In other terms, the Steiner ratio of a \( d \)-dimensional Einstein-Riemann space depends only from the dimension \( d \), and not from the specific choice of the matrix.
3.13 The Steiner Ratio of $L^d_p$

We will determine upper bounds for the Steiner ratio of $d$-dimensional $L_p$-spaces, abbreviated by $m(d, p)$, that is

$$m(d, p) = m(L^d_p),$$  \quad (3.68)

where $1 \leq p \leq \infty$ and $d$ a positive integer.

Let $\Delta_{i,j}$ be the Kronecker-symbol. Then a $d$-dimensional cross-polytope is the convex hull of

$$N = \{v = (x_1, \ldots, x_d) : x_{i,j} = \Delta_{i,j}, i, j = 1, \ldots, d\} \cup \{v = -v_{i-d} : i = d+1, \ldots, 2d\}$$

which contains $2d$ points. For $1 \leq i < j \leq 2d$ we have

$$\rho(v_i, v_j) = \begin{cases} 2 : & j = i + d \\ 2^{1/p} \leq 2 : & \text{otherwise} \end{cases}$$

and consequently

$$L(\text{MST for } N) = (2d - 1) \cdot 2^{1/p}.$$  

If we add the origin $o$, we find a shorter tree. More exactly,

$$L(\text{SMT for } N) \leq L(\text{MST for } N \cup \{o\}) = 2d,$$

using $\rho(v_i, o) = 1$ for $i = 1, \ldots, 2d$. Hence, it was proved

**Theorem 3.13.1** (Albrecht [1], [27]) For the Steiner ratio of the space $L^d_p$ it holds that

$$m(d, p) \leq \frac{2d}{2d-1} \cdot \left(\frac{1}{2}\right)^{1/p}.$$  

Obviously, the bound given in 3.13.1 is monotonically increasing in the value $p$. Hence, we may assume that for "big" $p$ we will find a better bound using the dual polytope of a cross-polytope. And indeed,

**Theorem 3.13.2** (Albrecht [1], [27]) For the Steiner ratio of the space $L^d_p$ it holds that

$$m(d, p) \leq \frac{2^{d-1}}{2^d - 1} \cdot d^{1/p}.$$  

**Proof.** Let $N$ be the set of the $2^d$ points $(\pm 1, \ldots, \pm 1)$. Then conv$N$ is a $d$-dimensional hypercube. The mutual distances between two different points in $N$ is
at least 2. It is not hard to see that an MST has length $2 \cdot (2^d - 1)$.

Let 
\[ T = (N \cup \{o\}, \{\overline{ov} : v \in N\}), \]
then it holds that
\[
m(d, p) \leq \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} \leq \frac{L(T)}{2(2^d - 1)} = \frac{2^d \cdot d^{1/p}}{2(2^d - 1)},
\]
using $\rho(o, v) = d^{1/p}$ for any $v \in N$.

\[ \square \]

Other than the bound given in 3.13.1, the bound given in 3.13.2 is monotonically decreasing in the value $p$. Hence, if $p$ runs to infinity, we have

**Corollary 3.13.3** It holds that
\[
m(d, \infty) \leq \frac{2^{d-1}}{2^d - 1}.
\]

Comparing 3.3.1 and 3.13.3 we have

<table>
<thead>
<tr>
<th>dimension $d$</th>
<th>$m(d, 1) \leq$</th>
<th>$m(d, \infty) \leq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.66666...</td>
<td>0.66666...</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>0.57142...</td>
</tr>
<tr>
<td>4</td>
<td>0.57142...</td>
<td>0.53333...</td>
</tr>
<tr>
<td>5</td>
<td>0.55555...</td>
<td>0.51612...</td>
</tr>
<tr>
<td>6</td>
<td>0.54545...</td>
<td>0.50793...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\to \infty$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

which says that $m(d, \infty)$ runs faster to $1/2$ than $m(d, 1)$.\[17\]:

**Conjecture 3.13.4** $c_d = m(d, \infty)$.

Note, that the conjectures 3.10.4 and 3.13.4 are independently, except we can show that there does not exists a Banach-Minkowski space with Steiner ratio 0.5.

\[ ^{17} \text{and moreover than } m(d, p) \text{ for } p > 1? \]
3.14 The Jung number

Saying that the Steiner ratio is a measure of the geometry of the space related to its combinatoric properties forces the interest of other measures.

We investigate quantities which are in relation to the distances in Banach-Minkowski spaces. Particularly, we are interested in the diameter of bounded sets and, moreover, in pairs of points in such sets which achieve this value.

For a bounded set $X$ in a Banach-Minkowski space $M_d(B)$, we define the diameter as

$$D_B(X) = \sup \{||v - v'||_B : v, v' \in X\}$$  \hspace{1cm} (3.69)

and the (circum-) radius as

$$R_B(X) = \inf \{r \geq 0 : v_0 \in A_d, v_0 + rB \supseteq X\}.$$  \hspace{1cm} (3.70)

(If the set $X$ is a compact set we have max and min.)

The value

$$J_d(B) = \sup \left\{ \frac{R_B(X)}{D_B(X)} : X \text{ is a bounded set in } M_d(B) \right\}$$  \hspace{1cm} (3.71)

is a geometrical constant, called the Jung number (of the space $M_d(B)$).

**Observation 3.14.1**

$$\frac{1}{2} \leq J_d(B) \leq \frac{d}{d+1},$$  \hspace{1cm} (3.72)

For a proof see [82].

With help of an easy calculation, we obtain the following result:

**Theorem 3.14.2** There are the following interrelations between the Jung number and the Steiner ratio of Banach-Minkowski spaces $M_d(B)$:

(a) $m_2(B) \leq \frac{2}{3} \cdot J_2(B)$.

(b) If there is a regular simplex with unit edge length in $M_d(B)$ then

$$m_d(B) \leq \left(1 + \frac{1}{d}\right) \cdot J_d(B).$$

Unfortunately, equality does not hold in general.
3.15 Equilateral sets

Of course, there is an equidistant set of $d + 1$ points in the Euclidean space $M_d(B(2))$, namely nodes of a regular simplex.\footnote{18}{Remember that we used this fact in the proof of 3.11.1.}

On the other hand, it is an open question whether there exist $d + 1$ equidistant points in any $d$-dimensional Banach-Minkowski space, even if the unit ball is smooth and if $d = 4$. Petty [94] shows that any set of equidistant points in a $d$-dimensional Banach-Minkowski space has at most the cardinality $2^d$, and equality is attained only when the unit ball is affinely equivalent to the $d$-dimensional hypercube. Also, for sufficiently large dimension $d$ in any $d$-dimensional affine space there exists a strictly convex unit ball $B$ such that there is an equidistant set in the space $M_d(B)$ with at least $(1.02)^d$ points. For all these facts compare [80] and [53].

We will use the idea of the existence of a regular simplex similar to in Euclidean spaces. For our investigations we have the following facts: Let $1 < p < \infty$ and $d \geq 3$. Then there are in $L^d_p$ at least $d + 1$ equidistant points. This can be seen with the following considerations: Consider $d$ points, with exactly one coordinate equal to 1, and all the others equal to 0; that is for $i = 1, \ldots, d$ let

$$ v_i = (x_{i,1}, \ldots, x_{i,d}) $$

with

$$ x_{i,j} = \begin{cases} 1 : & i = j \\ 0 : & \text{otherwise} \end{cases} $$

It is

$$ ||v_i - v_j|| = 2^{1/p} \tag{3.73} $$

for all $1 \leq i < j \leq d$.

For the point $v = (x, \ldots, x)$ it holds that

$$ ||v - v_i|| = ||v - v_j|| \tag{3.74} $$

for all $1 \leq i, j \leq d$.

To create $||v - v_i|| = 2^{1/p}$ the value $x$ has to fulfill the equation

$$ ((d - 1)|x|^p + |1 - x|^p)^{1/p} = 2^{1/p}. $$

This we can realize by the fact that the function $f : [0, 1] \to \mathbb{R}$ with

$$ f(x) = ((d - 1)x^p + (1 - x)^p)^{1/p} - 2^{1/p} $$

has exactly one zero in $[0, 1]$.

**Theorem 3.15.1** (Albrecht [1], [2]) Let $1 < p < \infty$ and $d \geq 3$. Then

$$ m(d, p) \leq \frac{d + 1}{2d} \cdot \left(\frac{d}{2}\right)^{1/p}. $$

18Remember that we used this fact in the proof of 3.11.1.
Extending this method we have

**Theorem 3.15.2** (Albrecht [1], [2]) Let $1 < p < \infty$. Then

$$m(d, p) < \frac{d + 1}{d} \cdot \left(\frac{1}{2}\right)^{1/p}.$$  

**Proof.** Let $N$ be the set with the $d + 1$ points constructed above and let $w$ be the "center" of this construction. Then

$$L(\text{MST for } N) = d$$
and

$$L(\text{SMT for } N) \leq (d + 1) \cdot 2^{-1/p}.$$  

These facts imply the assertion.  

This bound is not sharp, since the estimation of the distance of the points to the center is too inefficient, at least for small dimensions. On the other hand, we only use one additional point, and it is to be assumed that more than one of such points will decrease the length.

Now, we compare the bounds given in 3.15.1 and 3.15.2. Obviously,

$$\frac{d + 1}{2d} \cdot \left(\frac{d}{2}\right)^{1/p} \leq \frac{d + 1}{d} \cdot \left(\frac{1}{2}\right)^{1/p}$$  

(3.75)

holds if and only if

$$d \leq 2^p.$$  

(3.76)

Hence,

**Observation 3.15.3** Looking for the Steiner ratio of high dimensional $\mathcal{L}_p$-spaces we have only consider the bound given in 3.15.2, more exactly, when (3.76) is satisfied.

3.16 The Steiner ratio of $\mathcal{L}_{2k}^d$

It is obvious that all one-dimensional Banach spaces are isometric to each other so that $M_1(B(p))$ can be embedded into $M_{d'}(B(q))$ for any dimension $d'$ and for any real number $q \geq 1$. Also, it is clear that $M_d(B(p))$ can be embedded into $M_{d'}(B(p))$ for any $d' \geq d$ and any $p$. This, together with the theorem 3.3.5 implies

**Observation 3.16.1**

$$m(d, p) \geq m(d', p)$$  

(3.77)

for any number $p$ and for $d' \geq d$.

In other terms the function $m(d, p)$ is monotonically decreasing with respect to the dimension $d$.  

44
Banach [7] proved if \( p \neq 2 \) then each isometric embedding from a space \( M_d(B(p)) \) into itself is a permutation of the basis vectors followed by a sign change of some of these vectors.

Clearly, we are interested in the cases that \( d' > d \geq 2 \) and \( p \neq q \). Unfortunately, isometric embeddings are rare:

**Remark 3.16.2** For isometric embeddings between finite-dimensional \( \mathcal{L}_p \)-spaces the following holds true:

**(a)** (Lyubich, Vaserstein [86])

An isometric embedding \( M_d(B(\infty)) \rightarrow M_{d'}(B(q)) \) exists if and only if \( d = 2 \) and \( q = 1 \).

An isometric embedding \( M_d(B(p)) \rightarrow M_{d'}(B(\infty)) \) exists if and only if \( p = 1 \) and \( d' \geq 2^{d-1} \).

**(b)** (Lyubich, Vaserstein [87])

If \( p, q \neq \infty \) and there is an isometric embedding from \( M_d(B(p)) \) into \( M_{d'}(B(q)) \) then \( p = 2 \), and \( q \) is an even integer.

Next, we describe a consequence of 3.16.2(a) for the Steiner ratio. Let

\[
\phi : M_d(B(1)) \rightarrow M_d(B(\infty))
\]

be an isometric embedding. Then 3.16.2 and 3.3.1(a) obtain

\[
m_{d'}(B(\infty)) \leq m_d(B(1)) \leq \frac{d}{2d'-1}.
\]

With help of 3.16.2(a) and the monotonicity of the Steiner ratio we assume \( d' = 2^{d-1} \), i.e. \( d = \log_2 d' + 1 \). Consequently,

\[
m(d, \infty) \leq \frac{\log_2 d + 1}{2 \cdot \log_2 d + 1}.
\]

Hence, the Steiner ratio of \( M_d(B(\infty)) \) tends to \( 1/2 \) if the dimension \( d \) runs to infinity.\(^{19}\)

In general, it is not simple to construct isometric embeddings.\(^{20}\) Fortunately, there is a well-known mathematical question which needs these maps. The following isometric embeddings \( \phi : M_d(B(2)) \rightarrow M_{d'}(B(q)) \) are known in connection with Waring’s problem, which is a problem in number theory:

| J.Liouville: | \( d = 4 \) | \( d' = 12 \) | \( q = 4 \) |
| E.Lucas: | \( d = 3 \) | \( d' = 7 \) | \( q = 4 \) |
| A.Fleck: | \( d = 4 \) | \( d' = 32 \) | \( q = 6 \) |
| A.Hurwitz: | \( d = 4 \) | \( d' = 72 \) | \( q = 8 \) |
| I.Schur: | \( d = 4 \) | \( d' = 72 \) | \( q = 10 \) |

\(^{19}\)Please note that we can also obtain this fact by a simple calculation.

\(^{20}\)The remark 3.16.2(b) gives only a necessary condition.
compare König [77].

The above constructions of isometric embeddings should close this gap:

Suppose that \( q \) is an even integer. Let \( \phi : A_d \to A_{d'} \) with

\[
||v||_{B(2)} = ||\phi(v)||_{B(q)}
\]

for all vectors \( v \), be an isometric embedding from \( M_d(B(2)) \) into \( M_{d'}(B(q)) \).

Let \( \{e_i : i = 1, \ldots, d\} \) and \( \{f_j : j = 1, \ldots, d'\} \) be the standard bases for the spaces \( M_d(B(2)) \) and \( M_{d'}(B(q)) \), respectively. Using the standard inner product \( (\cdot, \cdot) \), we can represent each vector \( v \in A_d \) and \( w \in A_{d'} \) with respect to these bases. This is done as follows:

\[
v_i = (v, e_i)
\]

and

\[
w_j = (w, f_j) = (\phi(v), f_j) = (v, \phi^T(f_j)) =: (v, r_j),
\]

for \( i = 1, \ldots, d \) and \( j = 1, \ldots, d' \).

The system \( \{r_j : j = 1, \ldots, d'\} \) plays an important role. We call it the frame of the isometric embedding. The frame of the linear mapping consists of the rows of the standard matrix for \( \phi \).

In terms of coordinates, the condition for an isometric embedding reads as follows: A linear map is an isometric embedding if and only if

\[
(v, v)^{q/2} = \sum_{j=1}^{d'} (v, r_j)^q
\]

for all \( v \in A_d \) for its frame.

It is convenient to define the Waring number \( W(d, q) \) as follows:

\[
W(d, q) = \min\{d' \in \mathbb{N} : \text{there is an isometric embedding } \phi : M_d(B(2)) \to M_{d'}(B(q))\}
\]

That means that an isometric embedding \( M_d(B(2)) \to M_{d'}(B(q)) \) exists if and only if \( d' \geq W(d, q) \).

The Waring number \( W(d, q) \) is well-defined as a consequence of the proof by Hilbert and Stridsberg. Moreover, it follows that

\textbf{Remark 3.16.3} (Lyubich, Vaserstein [87]) For the Waring number the following are known, where \( q \) is an even integer:

\textbf{(a)} \( W(d, q) \) is monotone, which means

\[ W(d - 1, q) \leq W(d, q) \leq W(d, q + 2). \]

\textbf{(b)} \( W(2, q) = q/2 + 1 \).
W(d, q) grows exponentially in the dimension, more exactly, the inequalities
\[
\left(\frac{d + q/2 - 1}{d - 1}\right) \leq W(d, q) \leq \left(\frac{d + q - 1}{d - 1}\right).
\]
hold.

An exact value of \(W(d, q)\) is only known for small values of \(d\) and \(q\). König [77], Lyubich, Vasertien [87] and Seidel [101] have computed several Waring numbers exactly:

\[
\begin{align*}
W(3, 4) &= 6 \\
W(3, 6) &= 11 \\
W(3, 8) &= 16 \\
W(4, 4) &= 11 \\
W(7, 4) &= 28 \\
W(8, 6) &= 120 \\
W(23, 4) &= 276 \\
W(23, 6) &= 2300 \\
W(24, 10) &= 98280 \\
\end{align*}
\]

In view of the properties of the Waring number we obtain

**Theorem 3.16.4** (Cieslik [25]) For the Steiner ratio of \(\mathcal{L}_p^d\), where \(q\) is an even integer, we have

\[m(d, 2) \geq m(d', q)\]

for any dimension \(d' \geq W(d, q)\).

For instance, recalling the Waring numbers above, we see that the Steiner ratio for the \(d\)-dimensional \(\mathcal{L}_p\)-spaces can bounded in:

**Corollary 3.16.5** (Cieslik [25]) Using our knowledge about the Waring numbers we find the following bounds for the Steiner ratio of finite-dimensional \(\mathcal{L}_p\)-spaces.

(a) The Steiner ratio of \(\mathcal{L}_p^d\) has the following upper bounds:

\[
\begin{align*}
m(d, 4) &\leq 0.79280 \ldots \text{ for } d \geq 2; \\
m(d, 4) &\leq m(4, 2) \leq 0.76871 \ldots \text{ for } d > 10; \\
m(d, 4) &\leq m(7, 2) \leq 0.72247 \ldots \text{ for } d > 28; \\
m(d, 4) &\leq m(23, 2) \leq 0.69839 \ldots \text{ for } d > 275. \\
\end{align*}
\]

(b) The Steiner ratio of \(\mathcal{L}_6^d\) has the following upper bounds:

\[
\begin{align*}
m(d, 6) &\leq m(3, 2) \leq 0.78419 \ldots \text{ for } d > 10; \\
m(d, 6) &\leq m(8, 2) \leq 0.69455 \ldots \text{ for } d > 119; \\
m(d, 6) &\leq m(23, 2) \leq 0.69839 \ldots \text{ for } d > 2299. \\
\end{align*}
\]

(c) The Steiner ratio of \(\mathcal{L}_8^d\) has the following upper bounds:

\[
\begin{align*}
m(d, 8) &\leq m(3, 2) \leq 0.78419 \ldots \text{ for } d > 15. \\
\end{align*}
\]
(d) The Steiner ratio of $\mathcal{L}_d^{10}$ has the following upper bounds:

$$m(d, 10) \leq m(24, 2) \leq 0.69839 \ldots$$

for $d > 98279$.

In particular, $m(d, p)$ is a monotonically decreasing function in $d$. Since $m(d, p)$ is also bounded there exists the limit

$$m(p) = \lim_{d \to \infty} m(d, p).$$

The facts given above have the following consequences:

**Corollary 3.16.6** It holds that

$$m(p) = \lim_{d \to \infty} m(d, p) \leq 0.66984 \ldots$$

for any even integer $p$.

**Proof.** $W(d, q)$ increases in the dimension $d$, see 3.16.3(c) and (d). Consequently, if the even number $q$ is fixed then the Steiner ratio $m(d, q)$ tends to a limit less than or equal to the limit of $m(d, 2)$ which has been given in 3.11.2

□

3.17 $m(2, 4)$

As specification of our considerations above we find

**Theorem 3.17.1**

$$\sqrt{3} \cdot \sqrt{2} = 0.72823 \ldots \leq m(2, 4)$$

(3.85)

and

$$m(2, 4) \leq \frac{2}{3} \cdot \sqrt{2} = 0.79280 \ldots .$$

(3.86)

3.18 The Steiner Ratio for Banach-Minkowski Spaces of high Dimensions

There holds the following counterintuitive geometric assertion: Each unit ball in a sufficiently large dimensional Banach space has a large almost ellipsoidal section. More exactly, we use the Banach-Mazur distance, which is a natural similarity measure for two Banach spaces of the same dimension, in the following way: Let $B_d$ denote the class of all unit balls in $A_d$, and let $[B_d]$ be affine equivalence classes for $B_d$. Then the Banach-Mazur distance $\Delta$ is a metric on $[B_d]$ defined as

$$\Delta([B], [B']) = \inf\{h \geq 1 : \text{there is an bijective linear mapping } \Phi \text{ such that } B \subseteq \Phi B \subseteq hB\}$$

(3.87)

for $[B], [B']$ in $[B_d]$. 48
**Remark 3.18.1** (Dvoretzky [50]) For each positive real number $\epsilon$ and each positive integer $d'$ there is a number $D(\epsilon, d')$ such that every Banach-Minkowski space $M_d(B)$ of dimension $d$ at least $D(\epsilon, d')$ contains a $d'$-dimensional subspace $M_{d'}(B')$ such that

$$\Delta([B'], [B(2)]) \leq \ln(1 + \epsilon).$$

In terms of norms this fact means: For every positive integer $d'$ and every positive real $\epsilon$ there exists a number $D(\epsilon, d')$ such that for every norm $||.||$ in $A_d$, where $d \geq D(\epsilon, d')$, there exists a constant $c > 0$ and a subspace $A_{d'}$ such that

$$c \cdot ||v||_{\tilde{B}} \leq ||v|| \leq (1 + \epsilon) \cdot c \cdot ||v||_{\tilde{B}} \quad (3.88)$$

for all $v \in A_{d'}$, where $M_{d'}(\tilde{B})$ is isometric to the $d'$-dimensional Euclidean space.

Suppose that the assumption of remark 3.18.1 is satisfied and $M_{d'}(B')$ is the subspace of $M_d(B)$. Then we have,

$$m_d(B) \leq m_{d'}(B'). \quad (3.89)$$

Moreover, the inequality

$$\Delta([B'], [B(2)]) \leq \ln(1 + \epsilon) \quad (3.90)$$

implies (3.88). Then it is not hard to see that

$$m_{d'}(B') \leq (1 + \epsilon) \cdot m_{d'}(B(2)). \quad (3.91)$$

Both, (3.89) and (3.91) give the following

**Theorem 3.18.2** (Cieslik [30]) For the positive integer $d'$ and the positive real number $\epsilon$ let $D(\epsilon, d')$ be the Dvoretzky number, as defined in 3.18.1. Then for each Banach-Minkowski space $M_d(B)$ of dimension $d$ at least $D(\epsilon, d')$ the inequality

$$m_d(B) \leq (1 + \epsilon) \cdot m_{d'}(B(2))$$

holds.

### 3.19 When the dimension runs to infinity

Remember, that the quantity $C_d$ is defined as the upper bound of all numbers $m_d(B)$ ranging over all unit balls $B$ of the $d$-dimensional affine space $A_d$:

$$C_d = \sup \{m_d(B) : B \text{ a unit ball in } A_d\}. \quad (3.92)$$

Consider the sequence $\{C_d\}_{d=1,2,\ldots}$. In view of 3.3.5 and 2.2.1 this sequence, starting with $C_1 = 1$, is a decreasing and bounded, consequently a convergent one. 3.18.2 implies

$$m_d(B(2)) \leq C_d \leq (1 + \epsilon) \cdot m_{d'}(B(2)) \leq (1 + \epsilon) \cdot C_{d'},$$

if $d \geq D(\epsilon, d')$. Suppose that $d'$ runs to infinity, then $d$ does as well. Hence,
Theorem 3.19.1 (Cieslik [23]) Let the quantity $C_d$ defined as the upper bound of all numbers $m_d(B)$ ranging over all unit balls $B$ of the $d$-dimensional affine space. Then \( \{C_d\}_{d=1,2,...} \) is a decreasing and convergent sequence with
\[
\lim_{d \to \infty} C_d = \lim_{d \to \infty} m_d(B(2)).
\]

On the other hand, we are interested in
\[
c_d = \inf \{m_d(B) : B \text{ a unit ball in } A_d\}.
\]
Using 2.2.1 and 3.3.1 we have
\[
\frac{1}{2} \leq m_d(B(1)) \leq \frac{d}{2d-1}.
\]
Consequently,

Theorem 3.19.2 Let the quantity $c_d$ defined as the lower bound of all numbers $m_d(B)$ ranging over all unit balls $B$ of the $d$-dimensional affine space. Then \( \{c_d\}_{d=1,2,...} \) is a convergent sequence with
\[
\lim_{d \to \infty} c_d = \frac{1}{2}.
\]

3.20 The Steiner ratio of dual spaces

Let $B$ be a unit ball in the affine space and $DB$ its dual. Then it is often conjectured that they have equal Steiner ratios: $m_2(B) = m_2(DB)$. Wan et.al. [114] give a partial answer showing that this is true for sets with at most five points.

The relation between $m_d(B)$ and $m_d(DB)$ for $d > 2$ is still an open problem and has not been discussed before.

Conjecture 3.20.1 (Du, Lu, Ngo, Pardalos [47]) The Steiner ratio in any Banach-Minkowski space equals that in its dual space.

Maybe this conjecture is true in the planar case. But in higher dimensions we conjectured that the Steiner ratios are different. This conjecture is motivated by investigations of $L_p$-spaces, where in the plane we find similar behavior of the duals, but in higher-dimensional spaces there are several differences, for instance see the facts of the vertex-degrees in discussed in [21] or [22].

In 3.3.2 there is the conjecture that $m(3,1) = 3/5 = 0.6$, but in 3.13.3 we saw that $m(3, \infty) \leq 4/7 = 0.571 \ldots$. And general,

Theorem 3.20.2 Consider Banach-Minkowski spaces of a dimension $d$. For each $d \geq 3$, at least one of the conjectures 3.3.2 and 3.20.1 is false.
Proof. Assuming that both conjectures are true. Then

\[
\frac{d}{2d - 1} = m(d, 1) = m(d, \infty) \leq \frac{2^{d-1}}{2d - 1},
\]

(3.95)

using 3.13.3. For \(d \geq 3\) this is not a correct inequality.
Chapter 4

The Steiner ratio of Banach-Wiener Spaces

A Banach-Wiener space is an infinite-dimensional linear space equipped with a norm, which is a real-valued positive and homogenous function, which satisfied the triangle inequality; and makes the derived metric space complete.\textsuperscript{1} The structure of such spaces is intrinsically more complicated than that of the finite dimensional ones.

4.1 Steiner’s Problem in Banach-Wiener spaces

Now, we are interested in normed spaces which are not necessarily finite-dimensional. The idea of normed spaces is based on the same assumption of a norm than in the finite-dimensional case, namely that each vector of a space can be assigned its "length" or norm, which satisfies some "natural" conditions: positivity, identity, homogeneity and the triangle inequality.

The class of infinite-dimensional Banach spaces is more complicated that their of finite-dimensional ones. Here we have to define Steiner’s Problem more carefully: Remember that Steiner’s Problem is the "Problem of Shortest Connectivity". Since the demand of shortness forces the network to be cycle-less it is only necessary to consider trees.

Let $N$ be a finite set of points in the space $X$. For a given natural number $k$ and for $k$ points $v_1, \ldots, v_k \in X \setminus N$, let $T(k, v_1, \ldots, v_k)$ be a spanning tree of minimal length in the complete graph with the set $N \cup \{v_1, \ldots, v_k\}$ of vertices, where the length of the graph is induced by the metric.\textsuperscript{2}

\textsuperscript{1}For the name compare [117].

\textsuperscript{2}Remember, that we saw that a minimum spanning tree always exists and can be found easily in any metric space.
If there are both a number \( k' \) and points \( w_1, ..., w_{k'} \) such that the value 
\[ L(X)(T(k', w_1, ..., w_{k'})) \]
is minimal among all candidates \( T(k, v_1, ..., v_k) \), then we call \( T(k', w_1, ..., w_{k'}) \) a Steiner Minimal Tree (SMT) for \( N \), and the points \( w_1, ..., w_{k'} \) are called Steiner points. That means, an SMT for \( N \) is a minimum spanning tree on \( N \cup Q \), where \( Q \) is a set of additional vertices inserted into the metric space in order to achieve a minimal solution.

It is not true that there is an SMT for any given finite set in each metric space. Baronti, Casini and Papini [6] consider \( c_0 \), the usual space of (infinite) sequences of reals with supremum-norm. They show that there are three points in \( c_0 \) without a Torricelli point. In other terms, there are Banach spaces in which an SMT for specific finite sets does not exist. Of course, an MST in any case exists. Hence, we define the Steiner ratio more carefully in the following way:

\[
m(X) = \inf \left\{ \frac{L(SMT \text{ for } N)}{L(MST \text{ for } N)} : N \subseteq X \text{ a finite set for which an SMT exits} \right\}. \quad (4.1)
\]

To find the range of the Steiner ratio we recall that the proof of 2.2.1 does not use any specific property of a metric space. In particular, the dimension of the space is without interest. Hence,

**Theorem 4.1.1** The Steiner ratio of any Banach space is at least \( 1/2 \).

We will see that several spaces have the Steiner ratio \( 1/2 \), in particular space of infinite sequences, see below. Consequently,

**Corollary 4.1.2** This bound \( 1/2 \) is the best possible one over the class of all Banach spaces.

For the upper bound of the Steiner ratio we have

**Conjecture 4.1.3** (Du, Lu, Ngo, Pardalos [47]) The Steiner ratio in any infinite-dimensional Banach space \( X \) it holds

\[
m(X) \leq \frac{\sqrt{3}}{4 - \sqrt{2}} = 0.66983\ldots.
\]

Furthermore, we are interested in the quantity

\[
C_{\infty} = \sup \{ m(X) : X \text{ a Banach-Wiener space} \}
\]

(4.2)

**4.2 Isometric embeddings**

Assume that we know the Steiner ratio of a Banach space \( X' \) and we have that \( X' \) is a subspace of the Banach space \( X \). Then \( m(X) \) must be less or equal than \( m(X') \):

\[
m(X) \leq m(X'). \quad (4.3)
\]
The proof is similar to the proof of 3.3.5.

This observation is the core of the present chapter, but in a less weak form: We consider functions which map the space $X'$ into $X$ which preserve the distance between points. An isometry that maps the metric space $X'$ into a subspace of the space $X$ is called an isometric embedding of $X'$ into $X$. Each isometric embedding is an injective function. We have

**Theorem 4.2.1** Let be an isometric embedding from $X'$ into $X$ be given. Then $m(X') \geq m(X)$.

**Proof.** Let $N$ be a finite set in $X'$, and let $\phi : X' \to X$ be an isometric embedding. Then $\phi(N)$ is a finite set in $X$ with the following properties:

- $\phi(N)$ is a set of points in the image $\phi(X')$;
- $\phi(N)$ has the same cardinality as $N$: $|\phi(N)| = |N|$;
- The mutual distances between the points in $N$ and between the corresponding points in $\phi(N)$ are equal.

This implies the following equation:

$$L(X')(\text{MST for } N) = L(X)(\text{MST for } \phi(N)). \quad (4.4)$$

Moreover,

$$\phi(X') \subseteq X. \quad (4.5)$$

It is possible that an SMT for $\phi(N)$ in the space $X$ is shorter than in the subspace $\phi(X')$, but in any case

$$L(X')(\text{SMT for } N) \geq L(X)(\text{SMT for } \phi(N)) \quad (4.6)$$

holds.

Both, (4.4) and (4.6) imply the assertion for $\phi(N)$. Then the theorem follows in view of (4.5).

$\square$

Consequently,

**Corollary 4.2.2** Let $X$ be a Banach space. Then

$$m(X) \leq \inf \{m(X') : X' \text{ a subspace of } X\}. \quad (4.7)$$
4.3 Using Dvoretzky’s theorem for Banach-Wiener spaces

The Banach-Mazur distance between two not necessarily equal-dimensional Banach spaces $X$ and $Y$ can be defined more generally by:

$$
\Delta(X, Y) = \ln \inf \{ ||\Phi|| ||\Phi^{-1}|| : \Phi : X \to Y \text{ a isomorphism}\}.
$$

(4.8)

3.18.1 can generalized to

**Remark 4.3.1 (Dvoretzky [50])** Every infinite-dimensional Banach space $X$ contains the space $L_2^d$ almost isometrically, which means, that for every $\epsilon > 0$ and for every $d$ there is a unit ball $B(2)$ with

$$
\Delta(X, L_2^d) < 1 + \epsilon.
$$

(4.9)

In other terms: For every positive integer $d$ and every real $\epsilon > 0$ there is an operator $\Phi : L_2^d \to X$ such that

$$
||v|| \leq ||\Phi v|| \leq (1 + \epsilon) \cdot ||v||
$$

(4.10)

Similar to 3.18.2 and in view of (4.2.2) we get that the conjecture 4.1.3 is true. Moreover,

**Theorem 4.3.2 (Cieslik, Reisner [35])** Let $X$ be a infinite-dimensional Banach space, then

$$
0.5 \leq m(X) \leq \inf \{ m(d, 2) : d \text{ positive integer} \} = \lim_{d \to \infty} m(d, 2),
$$

where $m(d, 2)$ denotes the Steiner ratio of the $d$-dimensional Euclidean space.

4.4 A Banach-Wiener space with Steiner ratio 0.5

Consider the set $c_0$ of all convergent sequences with supremum norm

$$
||s|| = \sup \{|a_i| : i = 0, \ldots, \infty\}
$$

(4.11)

for $s = a_0, a_1, a_2, \ldots$.

Let $s_i$ be the sequence which consists of the real 0, except the $i$th position where the real 1 is located. Obviously,

$$
||s_i - s_j|| = \begin{cases} 1 & : i \neq j \\ 0 & : \text{otherwise} \end{cases}
$$

Now we investigate the set

$$
N(n) = \{s_0, \ldots, s_{n-1}\}
$$

(4.12)
of such sequences of $c_o$, and find immediately
\begin{equation}
L(\text{MST for } N(n)) = n - 1. \tag{4.13}
\end{equation}
Consider the sequence $s = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots$ such that
\begin{equation}
\|s_i - s\| = \frac{1}{2}
\end{equation}
for all numbers $i$, using as Steinerpoint we find
\begin{equation}
L(\text{SMT for } N(n)) \leq \frac{n}{2}. \tag{4.14}
\end{equation}
Thus the Steiner ratio of $c_o$ must be less or equal $n/2(n - 1)$, and this for all values of $n$. In other terms,
\begin{equation}
\text{Theorem 4.4.1 } m(c_0) = 0.5.
\end{equation}

### 4.5 The Steiner ratio of $l_p$

Consider the set $l_p$ of all sequences $s = \{a_k\}_{k=0,1,\ldots}$ where the norm

\begin{equation}
\|s\| = \left( \sum_{k=0}^{\infty} |a_k|^p \right)^{1/p}, \tag{4.15}
\end{equation}
$p \geq 1$, exists.

Consider $s_i$ be the sequence which consists of the real 0, except the $i$th position where the real 1 is located. Then,
\begin{equation}
\|s_i - s_j\| = \begin{cases} 2^{1/p} & : i \neq j \\ 0 & : \text{otherwise} \end{cases}
\end{equation}
Now we investigate the set
\begin{equation}
N(n) = \{s_0, \ldots, s_{n-1}\} \tag{4.16}
\end{equation}
of such sequences. We find
\begin{equation}
L(\text{MST for } N(n)) = (n - 1) \cdot 2^{1/p}. \tag{4.17}
\end{equation}
Consider the sequence $s = \{0\}_{k=0,1,\ldots}$ such that
\begin{equation}
\|s_i - s\| = 1
\end{equation}
for all numbers $i$. Using $s$ as Steinerpoint we find
\begin{equation}
L(\text{SMT for } N(n)) \leq n. \tag{4.18}
\end{equation}
Therefore,
\begin{equation}
m(l_p) \leq \frac{n}{(n - 1) \cdot 2^{1/p}}, \tag{4.19}
\end{equation}
for all $n$. If $n$ runs to infinity, we obtain the following bound.
Theorem 4.5.1

\[ m(l_p) \leq \left( \frac{1}{2} \right)^{1/p}, \]  

(4.20)

For \( p = 1 \) this is a tight bound, since

Corollary 4.5.2

\[ m(l_1) = \frac{1}{2} \]  

(4.21)

What can we say for the other values?

When we investigate the inequality

\[ \left( \frac{1}{2} \right)^{1/p} \leq C_{\infty}, \]  

(4.22)

we see that this equivalent to

\[ p \geq -\frac{\ln 2}{\ln C_{\infty}}. \]  

(4.23)

In view of 4.1.3 this is satisfied if

\[ p \geq 1.7328\ldots. \]

4.6 The range of the Steiner ratio

We saw that the Steiner ratio of Banach-Wiener spaces lies between 0.5 and 0.66983\ldots. But these are worst cases. What is the range of this quantity? Maybe, almost all Banach-Wiener spaces have Steiner ratio 1/2, or not?
Chapter 5

The Steiner ratio of metric spaces (cont.)

5.1 The ratio

Note, that there are metric spaces in which not any finite set has an SMT: Ivanov, Ryzhikow, Tuzhilin [72]: Let \( X \) be the set of all positive integers. A metric is defined by

\[
\rho(m, n) = \begin{cases} 
0 : & m = n \\
\frac{1}{m+n} + 1 : & m \neq n 
\end{cases}
\]

Then, consider the three-element set

\[ N = \{(0,0,0),(0,1,1),(1,0,1)\} \quad (5.1) \]

in the complete metric space

\[ (X^3, \tilde{\rho}) = \bigotimes_{i=1}^{3} (X, \rho). \quad (5.2) \]

The triangle spanned by \( N \) is equilateral, since the length of each of its sides equals 2. Hence, the length of an MST for \( N \) is 4. On the other hand, for any point \( q \not\in N \) we have \( \tilde{\rho}(v, q) > 1 \), therefore the length of an arbitrary tree constructed for \( N \cup \{q\} \) is strictly more than 3. But for \( q = (t, t, t), t > 1 \), we have

\[
\sum_{v \in N} \tilde{\rho}(v, q) = 3 + \frac{3}{t} \to 3
\]

when \( t \to \infty \). Thus, there does not exist an SMT for \( N \) in \( (X^3, \tilde{\rho}) \).

A complete description of all metric spaces in which Steiner’s Problem is solvable is not known and this situation is unlikely to change, because the class of all metric
space is too big. So it is necessary to prove the existence of an SMT for each specific metric space independently.

In view of this situation, we define the Steiner ratio by

\[ m(X) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \text{ a finite set in } X \text{ for which an SMT exists} \right\}. \]

5.2 The range of the Steiner ratio

Remember that the Steiner ratio of every metric space obeys

\[ m(X, \rho) \geq \frac{1}{2} = 0.5, \quad (5.3) \]

and this is the best possible bound.

Theorem 5.2.1 (Ivanov, Tuzhilin, [74])

a) For any real number between 0.5 and 1 there is a metric space with this Steiner ratio.

b) This remains true, restricting to finite spaces.

Sketch of the proof. Consider the metric space \( X = \{x_0, x_1, \ldots, x_n\} \) with

\[ \rho(x_i, x_j) = \begin{cases} 2 & : i, j \neq 0, i \neq j \\ a & : \text{otherwise} \end{cases} \]

where \( a \) is a variable real number, but with the following constraints:

1. Since \( \rho \) should be a metric we have \( 2 \leq a + a \), hence

\[ 1 \leq a. \]

2. An MST for \( N = \{x_1, \ldots, x_n\} \) has length \( 2(n - 1) \). A shorter tree is given insofar the star with center \( x_0 \) has length \( na \). Hence \( na < 2(n - 1) \) forced

\[ a \leq 2 - \frac{2}{n}. \]

There are metric spaces with Steiner ratio 1 and 1/2. For these extreme values we know:

- There are many metric spaces with Steiner ratio 1.
- There are infinite metric spaces of the Steiner ratio 1/2, but not a finite one.

Consequently, we have the complete intervall from 1/2 to 1 as the values for the Steiner ratio of metric spaces.
5.3 Several Properties

In the present section we give several facts which will be helpful for further considerations.

We need the following two Lemmas proved for the case of Banach-Minkowski spaces only, but the proof in the general case of metric spaces is just the same.

**Lemma 5.3.1** Let $X$ be a set, and $\rho_1$ and $\rho_2$ be two metrics on $X$. We assume that for some numbers $c_2 \geq c_1 > 0$ and for arbitrary points $x$ and $y$ from $X$ the following inequality holds:

$$c_1 \cdot \rho_2(x, y) \leq \rho_1(x, y) \leq c_2 \cdot \rho_2(x, y).$$  \hspace{1cm} (5.4)

Then

$$\frac{c_1}{c_2} \cdot m(X, \rho_2) \leq m(X, \rho_1) \leq \frac{c_2}{c_1} \cdot m(X, \rho_2).$$  \hspace{1cm} (5.5)

And, let $(X, \rho)$ be a metric space, and $Y \subset X$ be some of its subspace. Recall that Kruskal’s method, which finds an MST, uses only the mutual distances between the points. Hence, it holds that

$$L(Y, \rho)(\text{MST for } N) = L(X, \rho)(\text{MST for } N)$$

for any finite set $N$ of points in $Y$. On the other hand, it is possible that an SMT for $N$ in the space $(X, \rho)$ is shorter than in the subspace $(Y, \rho)$. Consequently, it holds that

$$L(X, \rho)(\text{SMT for } N) \leq L(Y, \rho)(\text{SMT for } N)$$

for any finite set $N$ of points in $Y$. So we have:

**Lemma 5.3.2** Let $(X, \rho)$ be a metric space, and $Y \subset X$ be some of its subspace. Then

$$m(Y, \rho) \geq m(X, \rho).$$  \hspace{1cm} (5.6)

The following proposition holds.

**Lemma 5.3.3** Let $f : X \rightarrow Y$ be some mapping of a metric space $(X, \rho_X)$ onto a metric space $(Y, \rho_Y)$. We assume that $f$ does not increase the distances, that is, for arbitrary points $x$ and $y$ from $X$ the following inequality holds:

$$\rho_Y(f(x), f(y)) \leq \rho_X(x, y).$$  \hspace{1cm} (5.7)

Then for arbitrary finite set $N \subset Y$ we have:

$$L(X)(\text{MST for } N) \geq L(Y)(\text{MST for } f(N)) \quad \text{and}$$

$$L(X)(\text{SMT for } N) \geq L(Y)(\text{SMT for } f(N)).$$  \hspace{1cm} (5.8)
Proof. Let $G$ be an arbitrary connected graph constructed on $N$. We consider two weight functions on $G$ defined on the edges $xy$ of $G$ as follows:

$$
\omega_Y (x, y) = \rho_Y (f(x), f(y)).
$$

Since $f$ does not increase the distances, it follows

$$
L(X)(G) \geq \omega_Y (G).
$$

Let $G'$ be a graph on $N' = f(N)$, such that the number of edges joining the vertices $x'$ and $y'$ from $N' = V(G')$ is equal to the number of edges from $G$ joining the vertices from $f^{-1}(x') \cap N$ with the vertices from $f^{-1}(y') \cap N$. It is clear that $G'$ is connected, and

$$
L(Y)(G') = \omega_Y (G).
$$

Conversely, it is easy to see that for an arbitrary connected graph $G'$ constructed on $f(N)$ there exists a connected graph $G_X$ on $N$, such that

$$
L(Y)(G') = \omega_Y (G_X).
$$

To construct $G_X$ it suffices to span each set $N \cap f^{-1}(x')$, $x' \in N'$, by a connected graph, and then to join each pair of the constructed graphs corresponding to some adjacent vertices $G'$ by $k$ edges, where $k$ is the multiplicity of the corresponding edge in $G'$.

Therefore,

$$
L(X)(MST \text{ for } N) = \inf \{ L(X)(G) : V(G) = N \}
\geq \inf \{ \omega_Y (G) : V(G) = N \}
= \inf \{ L(Y)(G') : V(G') = f(N) \}
= L(Y)(\text{MST for } f(N)).
$$

Thereby, the first inequality is proved.

Now let us prove the second inequality. We have:

$$
L(X)(SMT \text{ for } N) = \inf \{ L(X)(\text{MST for } \tilde{N}) : \tilde{N} \supset N \}
\geq \inf \{ L(Y)(\text{MST for } f(\tilde{N})) : \tilde{N} \supset N \}
\geq \inf \{ L(Y)(\text{MST for } \tilde{N}' : \tilde{N}' \supset f(N) \}
= L(Y)(\text{SMT for } f(N)).
$$

This lemma gives two theorems:

**Theorem 5.3.4** Let $f X \rightarrow Y$ be a mapping of a metric space $(X, \rho_X)$ to a metric space $(Y, \rho_Y)$, and let $f$ do not increase the distances. We assume that for each finite subset $N' \subseteq Y$ there exists a finite subset $N \subseteq X$, such that $f(N) = N'$ and

$$
L(X)(\text{SMT for } N) \leq L(Y)(\text{SMT for } N').
$$

(5.10)
Then
\[ m(X, \rho_X) \leq m(Y, \rho_Y). \]  
(5.11)

Theorem 5.3.4 can be slightly reinforced as follows.

**Theorem 5.3.5** Let \( f : X \to Y \) be a mapping of a metric space \((X, \rho_X)\) to a metric space \((Y, \rho_Y)\), and let \( f \) do not increase the distances. We assume that for each finite subset \( N' \subseteq Y \) the following inequality holds:
\[ \inf \{ L(X)(SMT \text{ for } N) : f(N) = N' \} \leq L(Y)(SMT \text{ for } N'). \]  
(5.12)

Then
\[ m(X, \rho_X) \leq m(Y, \rho_Y). \]  
(5.13)

### 5.4 The Steiner ratio of finite metric spaces

For a finite set \( X \) the space \( \mathbb{R}^X \) is a \(|X|\)-dimensional affine space.

**Lemma 5.4.1** Any finite metric space \((X, \rho)\) can be embedded in \( L_\infty^{|X|} \). Consequently,
\[ m(X, \rho) \geq m(|X|, \infty). \]  
(5.14)

We assume that conjecture 3.3.2 is true. Therefore,

**Theorem 5.4.2** For any finite metric space \((X, \rho)\) it holds
\[ m(X, \rho) \geq \frac{2^{|X|} - 1}{2^{|X|} - 1}. \]  
(5.15)

Hence, we find again,

**Corollary 5.4.3** No finite metric space has a Steiner ratio \( 1/2 \).

A little collection for these quantities is given by Cieslik [29].

### 5.5 The Steiner ratio of graphs

Each network \( G = (V, E) \) with length-function \( f : E \to \mathbb{R} \) is a metric space \((V, \rho)\) by defining the distance function in the way that \( \rho(v, v') \) is the length of a shortest path between the vertices \( v \) and \( v' \) in \( G \).

If there does not exist a length-function explicitly, we assume \( f \equiv 1 \), that means the distance \( \rho(v, v') \) is defined as the minimal number of edges connecting the vertices \( v \) and \( v' \) by a path in \( G \). A survey about graphs as metric spaces is presented in [121].

In this sense, we construct the so-called metric closure \( G_f \) defined as the complete graph on \( V \) such that the length of an edge \( vv' \) in \( G_f \) is the length of a shortest path between \( v \) and \( v' \) in \( G \). Using Dijkstra's algorithm \( G_f \) can be found in polynomially bounded time:
Algorithm 5.5.1 (Dijkstra [37]) Let $G = (V, E, f)$ be a network. A shortest path between the vertices $v$ and $v'$ can be found by the following procedure:

1. Start with the vertex $v$; Label $v$ with $0$: $L(v) := 0$; all other vertices are unlabelled;

2. Determine $\min\{L(v_1) + f(v_1, v_2)\}$ where $v_1$ and $v_2$ are adjacent vertices, $v_1$ labelled and $v_2$ not;
   Choose $\tilde{v}_1$ and $\tilde{v}_2$ which attain the minimum;
   Label $\tilde{v}_2$ by $L(\tilde{v}_2) = L(\tilde{v}_1) + f(\tilde{v}_1, \tilde{v}_2)$;

3. Repeat the second step until $v'$ is labelled.

For all labelled vertices $w$ the quantity $L(w)$ is the length of a shortest path connecting $v$ and $w$:
$$\rho(v, w) = L(w).$$

On the other hand, each finite metric space is a desired chosen finite graph; more
exactly:

Observation 5.5.2 (Hakimi, Yau [62]) Each finite metric space can be represented as a finite graph with a (nonnegative) length-function.

Proof. Let $(X, \rho)$ be a finite metric space. We define the graph $G = (X, E)$ as the complete graph on the vertex-set $X$. The length-function $f$ is given by the metric $\rho$.

In other terms, in graphs we obtain all finite metric spaces.

The Steiner ratio is of the form
$$m = m(G) = \min \left\{ \frac{L(SMT \ for \ N)}{L(MST \ for \ N)} : N \subseteq V \right\}. \quad (5.16)$$

In other terms,
$$m = m(G^f) = m(V, \rho), \quad (5.17)$$
where $G^f$ denotes the metric closure of the the graph $G$ with length-function $f$.

Let $S_k$ be a star with $k$ leaves. Considering the leaves as the set of given points we find an MST of length $2 \cdot (k - 1)$ and an SMT of length $k$. Hence,
$$m(S_k) \leq \frac{k}{2 \cdot (k - 1)} = \frac{1}{2 - \frac{2}{k}}. \quad (5.18)$$
This upper bound tends to $\frac{1}{2}$ if the number of leaves runs to infinity. Thus we have proved
Theorem 5.5.3 Let $G$ be a (connected) graph. Then for the Steiner ratio of $G$

$$\frac{1}{2} \leq m(G) \leq 1$$

holds.
These bounds are the best possible ones.

Now, we give a little collection of known Steiner ratios for (connected) graphs:

Theorem 5.5.4 The value for the Steiner ratio of complete graphs, paths and cycles equals 1.

Theorem 5.5.5 Let $G$ be a star with $k$ leaves, $k \geq 2$. Then

$$m(G) = \frac{k}{2 \cdot (k - 1)}.$$

Proof. Considering, the leaves as the set of given points we find an MST of length $2 \cdot (k - 1)$ and an SMT of length $k$. Hence,

$$m(G) \leq \frac{k}{2 \cdot (k - 1)} = \frac{1}{2 - \frac{2}{k}}. \quad (5.19)$$

It is easy to see that all other sets of given points do not give a smaller value of the Steiner ratio.

5.6 The Steiner ratio of ultrametric spaces

Up to now we have found in each space that the determination of an SMT is a hard problem. In the next example, we describe a class of metric spaces in which Steiner’s Problem is as easy as finding a minimum spanning tree.

Let $(X, \rho)$ be a metric space. $\rho$ is called an ultrametric if

$$\rho(v, w) \leq \max\{\rho(v, u), \rho(w, u)\} \quad (5.20)$$

for any points $u, v, w$ in $X$.

It is not hard to see that we have

Lemma 5.6.1 The following is true for any three points $u, v$ and $w$ in an ultrametric space $(X, \rho)$:

If $\rho(v, u) \neq \rho(w, u)$, then $\rho(x, y) = \max\{\rho(v, u), \rho(w, u)\}$.

That means that all triangles in $(X, \rho)$ are isosceles triangles where the base is the shorter side.
Let $T = (V, E)$ be an SMT for $N$. Let $Q$ denote the set of all Steiner points in $T$, i.e., $Q = V \setminus N$. Suppose that $Q$ is nonempty. There is a Steiner point $q$ in $Q$ such that $q$ is adjacent to two vertices $v$ and $v'$ in $N$. Using 5.6.1, we may assume that $\rho(v, v') = \rho(v, q)$. The tree

$$T' = (V, E \setminus \{vq\} \cup \{vv'\})$$

has the same length as $T$, and it is an SMT for $N$ too. If $g_{T'}(q) \geq 3$ we repeat this procedure. If $g_{T'}(q) = 2$ we find an SMT with a smaller number of Steiner points than $T$, since no Steiner point has degree smaller than 2.

Hence, we proved, that Steiner’s Problem in an ultrametric space is the same as finding an MST. Consequently,

**Observation 5.6.2** *The Steiner Ratio of an ultrametric space equals one.*

The converse statement is not true, since the real line has the Steiner ratio 1, but is not a ultrametric space.

An interesting question: What does the equality $m(X, \rho) = 1$ for a metric space $(X, \rho)$ mean? Note, that we will find several other metric spaces which Steiner ratio equals 1. Can we classify all spaces with Steiner ratio 1?

### 5.7 The Steiner ratio of Hamming spaces

We consider sequence spaces. For a word $v \in \{0, 1\}^d$ we define the Hamming weight $\text{wt}(v)$ as the number of times the digit "1" occurs in $v$.

Let $v$ and $w$ be words over $\{0, 1\}$ of length $d$. We define the Hamming distance by

$$\rho_H(v, w) = \text{wt}(v + w) = \text{wt}(v - w). \quad (5.21)$$

Conversely,

$$\text{wt}(v) = \rho_H(v, o), \quad (5.22)$$

where $o = 0^n$.

The Hamming distance between $v$ and $w$ is the number of positions in which $v$ and $w$ disagree.

It can be directly generalized to words in $A^d$, for an alphabet $A$:

$$\rho_H((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = |\{i : a_i \neq b_i \text{ for } i = 1, \ldots, d\}|, \quad (5.23)$$

for $a_i, b_i \in A$.

**Theorem 5.7.1**

$$\frac{1}{2} \leq m(A^d, \rho_H) \leq \frac{d}{2(d - 1)}, \quad (5.24)$$

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Consequently,

\[ m(A^d, \rho_H) \approx \frac{1}{2} \tag{5.25} \]

if \( d \gg 1 \), see Foulds [52].

An interesting observation: Let \( \mathcal{L}_1^d \) be the \( d \)-dimensional affine space with rectilinear distance, and let \( (\{0,1\}^d, \rho_H) \) be the space of sequences of length \( d \) over \( \{0,1\} \) with the Hamming distance. Two facts are easy to see:

- \( (\{0,1\}^d, \rho_H) \) is a subspace of \( \mathcal{L}_1^d \), and
- The Steiner ratio of \( (\{0,1\}^d, \rho_H) \) is less than or equal to \( d/(2d - 2) \).

Hence, by 5.3.2:

**Example 5.7.2**

\[ m(\mathcal{L}_1^d) \leq m(\{0,1\}^d, \rho_H) \leq \frac{d}{2(d-1)}. \]

On the other hand, we saw that

\[ m(\mathcal{L}_1^d) \leq \frac{d}{2d-1}, \]

which is a stronger result.

### 5.8 The Steiner ratio of phylogenetic spaces

We determine the Steiner ratio of Phylogenetic spaces. Consider an alphabet \( A \) with at least two letters \( a \) and \( b \), and use the Levenshtein distance, where the Levenshtein (or edit distance), between two words of not necessarily equal length is the minimal number of "edit operations" required to change one word into the other, where an edit operation is a deletion, insertion, or substitution of a single letter in either word.\(^1\)

To extend the Hamming distance to a metric for all words we may use the following way: Let \( A \) be a set of letters. Add a "dummy" letter "-" to \( A \). We define a map

\[ cl : (A \cup \{-\})^* \rightarrow A^* \tag{5.26} \]

deleting all dummies in a word from \( (A \cup \{-\})^* \). Then for two words \( w \) and \( w' \) in \( A^* \) we define the extended Hamming-distance as

\[
\rho(w, w') = \min \{ \rho_H(w, w') : w, w' \in (A \cup \{-\})^*, |w| = |w'|, cl(w) = w, cl(w') = w' \}. \tag{5.27}
\]

\(^1\)At first glance, it seems that the sequence spaces are subspaces of the phylogenetic space, but this is not true: Consider the two words \( v = (ab)^d \) and \( w = (ba)^d \); then \( \rho_L(v, w) = 2 \) but \( \rho_H(v, w) = 2d \).
The extended Hamming-distance coincides with the Levenshtein metric.
for a generalization of the Levenshtein distance see [32].

Consider the words $w_i$ which consist of the letter $a$ repeated $d$ times, except the $i$-th position where another letter $b$ is located, $i = 1, \ldots, d$. Then define the set

$$N(d) = \{w_i : |w_i| = d, i = 1, \ldots, d\}$$

(5.28)
of $d$ points.
For $i \neq j$ it holds that $\rho_L(w_i, w_j) = 2$. Hence,

$$L(\text{MST for } N(d)) = 2(d - 1).$$

(5.29)
The word $w = a \ldots a$ has distance 1 to any $w_i$. Consequently, the star with the center $w$ and the leaves $w_i, i = 1, \ldots, d$ is an SMT for $N(d)$ for which

$$L(\text{SMT for } N(d)) = d.$$ 

(5.30)
Both equations (5.29) and (5.30) give

$$m(A^*, \rho_L) \leq \frac{d}{2(d - 1)},$$

(5.31)
for all positive integers $d \geq 2$. Now, we have found a metric space which achieves the lower bound 0.5 for the Steiner ratio:

**Theorem 5.8.1** For the Steiner ratio of the phylogenetic space $(A^*, \rho_L), |A| \geq 2$, it holds that

$$m(A^*, \rho_L) = \frac{1}{2}.$$ 

(5.32)

Note that we don’t have a finite set $N_0$ of points such that

$$\mu(N_0) = \frac{L(\text{SMT for } N_0)}{L(\text{MST for } N_0)} = \frac{1}{2},$$

(5.33)
And, in view of 2.3.1, we cannot find such set.
Chapter 6

The Steiner ratio of manifolds

6.1 The Steiner ratio on spheres

Let $X$ be the surface of a Euclidean ball, called a sphere. A metric on $X$ is given by the shortest great circle distance between the points. Network minimization problems on $X$ are the so-called Large Region Location Problems. A general solution method for Steiner’s problem is still unknown except for some special cases, see [83] and [98].

**Theorem 6.1.1** (Rubinstein, Weng [98]) The Steiner ratio for spheres is the same as in the Euclidean plane.

*Proof idea.* Suppose that $\triangle u_1v_1w_1$ and $\triangle u_2v_2w_2$ are two triangles of equal side lengths lying on a sphere $\Sigma_i$, $i = 1, 2$ with radii $r_1 < r_2$ respectively. Then it will prove the existence of a map $h : \triangle u_1v_1w_1 \rightarrow \triangle u_2v_2w_2$ such that for any two points $p_1q_1 \in \triangle u_1v_1w_1$ it holds that

$$\rho(p_1, q_1) \geq \rho(h(p_1), h(q_1)).$$

(6.1)

Moreover, if $p_1$ and $q_1$ are not on the same side, then the inequality is strict. This compression theorem can be applied to compare the minimum of a variable in triangles on two spheres. Then the above assertion follows.

It seems that the proof needs similar methods than the proof of the Gilbert-Pollak-conjecture given by 3.5.1. Does this create the same gap?

6.2 Riemannian metrics

Let $M$ be an arbitrary connected $d$-dimensional Riemannian manifold. For each piecewise-smooth curve $\gamma$ by $\text{length}(\gamma)$ we denote the length of $\gamma$ with respect to the
Riemannian metric. By $\rho$ we denote the intrinsic metric generated by the Riemannian metric. We recall that

$$\rho(x, y) = \inf_{\gamma} \text{length}(\gamma),$$  \hspace{1cm} (6.2)

where the greatest lower bound is taken over all piecewise-smooth curves $\gamma$ joining the points $x$ and $y$.

Let $p$ be a point from $M$. We consider the normal coordinates $(x^1, \ldots, x^d)$ centered at $p$, such that the Riemannian metric $g_{ij}(x)$ calculated at $p$ coincides with $\delta_{ij}$. Let $U(\delta)$ be the open convex ball centered at $P$ and having the radius $\delta$. Any two points $x$ and $y$ from the ball are joined by a unique geodesic $\gamma$ lying in $U(\delta)$. At that time,

$$\rho(x, y) = \text{length}(\gamma).$$

Thus, the ball $U(\delta)$ is a metric space with intrinsic metric, that is, the distance between the points equals to the greatest lower bound of the curves' lengths over all the measurable curves joining the points. Notice that in terms of the coordinates $(x^i)$ the ball $U(\delta)$ is defined as follows:

$$U(\delta) = \{(x^1)^2 + \cdots + (x^d)^2 < \delta^2\}. \hspace{1cm} (6.3)$$

Therefore, if we define the Euclidean distance $\rho_e$ in $U(\delta)$ (in terms of the normal coordinates $(x^i)$), then the metric space $(U(\delta), \rho_e)$ also is the space with intrinsic metric generated by the Euclidean metric $\delta_{ij}$.

Since the Riemannian metric $g_{ij}(x)$ depends on $x \in U(\epsilon)$ smoothly, then for any $\epsilon$, $1/d^2 > \epsilon > 0$, there exists a $\delta > 0$, such that

$$|g_{ij}(x) - \delta_{ij}| < \epsilon \hspace{1cm} (6.4)$$

for all points $x \in U(\delta)$. The latter implies the following Proposition.

**Lemma 6.2.1** Let $\|v\|_g$ be the length of the tangent vector $v \in T_xM$ with respect to the Riemannian metric $g_{ij}$, and let $\|v\|_e$ be the length of $v$ with respect to the Euclidean metric $\delta_{ij}$. If for any $i$ and $j$ the inequality (6.4) holds, then

$$\sqrt{1 - d^2 \epsilon} \cdot \|v\|_e \leq \|v\|_g \leq \sqrt{1 + d^2 \epsilon} \cdot \|v\|_e. \hspace{1cm} (6.5)$$

Using the definition of the distance between a pair of points of a connected Riemannian manifold, we obtain the following result.

**Lemma 6.2.2** Let $M$ be an arbitrary connected $n$-dimensional Riemannian manifold, and let $U(\delta)$, $\rho$, and $\rho_e$ be as above. Then for an arbitrary $\epsilon$, $1/d^2 > \epsilon > 0$, there exists a $\delta > 0$, such that

$$\sqrt{1 - d^2 \epsilon} \cdot \rho_e(x, y) \leq \rho(x, y) \leq \sqrt{1 + d^2 \epsilon} \cdot \rho_e(x, y) \hspace{1cm} (6.6)$$

for all points $x, y \in U(\delta)$. 69
6.3 Riemannian manifolds

Since the Steiner ratio is evidently the same for any convex open subsets of $\mathbb{R}^d$, 6.2.2 and 5.3.1 lead to the following result.

**Corollary 6.3.1** Let $M$ be an arbitrary $d$-dimensional Riemannian manifold, let $U(\epsilon) \subseteq M$ be an open convex ball of a small radius $\epsilon$, and let $P$ be the center of $U(\epsilon)$. By $\rho$ we denote the metric on $M$ generated by the Riemannian metric. Then

$$\sqrt{\frac{1 - d^2 \epsilon}{1 + d^2 \epsilon}} \cdot m(\mathbb{R}^d) \leq m(U(\epsilon), \rho) \leq \sqrt{\frac{1 + d^2 \epsilon}{1 - d^2 \epsilon}} \cdot m(\mathbb{R}^d),$$

(6.7)

where $m(\mathbb{R}^n)$ stands for the Steiner ratio of the Euclidean space $\mathbb{R}^n$.

**Theorem 6.3.2** (Ivanov et al. [75]) The Steiner ratio of an arbitrary $d$-dimensional connected Riemannian manifold $M$ does not exceed the Steiner ratio of $\mathbb{R}^d$.

**Sketch of the proof.** 6.3.1 implies that

$$m(X_i, \rho) \leq \sqrt{\frac{1 + d^2 \epsilon}{1 - d^2 \epsilon}} \cdot m(\mathbb{R}^d).$$

(6.8)

Since $\sqrt{\frac{1 + d^2 \epsilon}{1 - d^2 \epsilon}} \to 1$ as $i \to \infty$ due to the choice of $\{\epsilon_i\}$, we get

$$\inf_i m(X_i, \rho) \leq m(\mathbb{R}^d).$$

(6.9)

But, due to 5.3.2 we have:

$$m(M, \rho) \leq \inf_i m(X_i, \rho).$$

(6.10)

Applying Proposition 5.3.5 gives:

**Theorem 6.3.3** (Ivanov et al. [75]) Let $\pi W \to M$ be a locally isometric covering of connected Riemannian manifolds. Then the Steiner ratio of the base $M$ of the covering is more or equal than the Steiner ratio of the total space $W$.

**Corollary 6.3.4** Assume that 3.5.1 is true. The Steiner ratio for a flat two-dimensional torus, a flat Klein bottle, a projective plain having constant positive curvature is equal to $\sqrt{3}/2$.

**Idea of the proof.** It follows from Theorems 6.3.2, and 6.3.3. Du and Hwang theorem [41] and [39] saying that the Steiner ratio of the Euclidean plane equals $\sqrt{3}/2$; and also from Rubinstein and Weng theorem [98] saying that the Steiner ratio of the standard two dimensional sphere with constant positive curvature metric equals $\sqrt{3}/2$.

Thus, taking into account the results of Rubinstein and Weng [98], the Steiner ratio is computed now for all closed surfaces having non-negative curvature.
6.4 Lobachevsky spaces

Let us consider the Poincare model of the Lobachevsky plane $L^2(-1)$ with constant curvature $-1$. We recall that this model is a radius 1 flat disk centered at the origin of the Euclidean plane with Cartesian coordinates $(x, y)$, and the metric $ds^2$ in the disk is defined as follows:

$$ds^2 = 4 \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}. \quad (6.11)$$

It is well known that for each regular triangle in the Lobachevsky plane the circumscribed circle exists. The radii emitted out of the center of the circle to the vertices of the triangle forms the angles of $120^\circ$.

Let $r$ be the radius of the circumscribed circle. The cosine rule implies that the length $a$ of the side of the regular triangle can be calculated as follows:

$$\cosh a = \cosh^2 r - \sinh^2 r \cos \frac{2\pi}{3} = 1 + \frac{3}{2} \sinh^2 r.$$

It is easy to verify that for such triangle the length of MST equals $2a$, and the length of SMT equals $3r$. Therefore, the Steiner ratio $m(r)$ for the regular triangle inscribed into the circle of radius $r$ in the Lobachevsky plane $L^2(-1)$ has the form

$$m(r) = \frac{3}{2} \cdot \frac{r}{\text{arccosh}(1 + \frac{3}{2} \sinh^2(r))}.$$ 

It is easy to calculate that limit of the function $m(r)$ as $r \to \infty$ is equal to $3/4$. Consequently,

**Theorem 6.4.1** (Ivanov et al. [76]) The Steiner ratio of the curvature $-1$ Lobachevsky space does not exceed $3/4$.

**Theorem 6.4.2** (Ivanov et al. [76]) The Steiner ratio of an arbitrary surface of constant negative curvature $-1$ is strictly less than $\sqrt{3}/2$.

**Proof.** It is easy to see that the Taylor series for the function $m(r)$ at $r = 0$ has the following form:

$$\frac{\sqrt{3}}{2} - \frac{r^2}{16\sqrt{3}} + O(r^4).$$

Therefore, $m(r)$ is strictly less than $\sqrt{3}/2$ in some interval $(0, \epsilon)$. The latter means that for sufficiently small regular triangles on the surfaces of constant curvature $-1$, the relation of the lengths of SMT and MST is strictly less than $\sqrt{3}/2$.

These results has been enforced for specific spaces.

**Theorem 6.4.3** (Innami, Kim [69]) The Steiner ratio of a simply connected manifold of negative constant curvature without boundary equals $1/2$. 

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Idea of the proof. First we use that the Steiner ratio is in any case at least $1/2$. On the other hand, we use 2.2.4, that means sets with many points. More exactly, we consider the Poincare disk, namely,

$$H = \{(x, y) : x^2 + y^2 < 1\}$$

with the Riemannian metric

$$ds^2 = 4 \frac{dx^2 + dy^2}{c(1 - x^2 - y^2)^2},$$

for a positive $c$. Any complete simply connected manifold of negative constant curvature $-c$ without boundary is isometric to $H$.

Let $n$ be an integer greater than 2. Let $O$ be the origin in $H$ and geodesic rays for $i = 1, \ldots, n$ such that

$$\gamma_i(0) = O,$$

$$\text{angle of } (\gamma_i'(0), \gamma_{i+1}'(0)) = \frac{2\pi}{n},$$

$$\gamma_{n+1} = \gamma_1.$$

Let

$$N(s) = \{\gamma_i(s) : i = 1, \ldots, n\}$$

for a positive $s$. $T(\gamma_i(s), \gamma_{i+1}(s))$ denotes the minimal subtree from $\gamma_i(s)$ to $\gamma_{i+1}(s)$ in the SMT of $N(s)$. Then it holds

$$\lim_{s \to \infty} \frac{L(T(\gamma_i(s), \gamma_{i+1}(s)))}{d(\gamma_i(s), \gamma_{i+1}(s))} = 1.$$  

(6.13)

By the choice of $N(s)$ we have

$$L(\text{MST for } (N(s))) = (n - 1)d(\gamma_1(s), \gamma_2(s)).$$  

(6.14)

Consequently,

$$\frac{L(\text{SMT for } (N(s)))}{L(\text{MST for } (N(s)))} = \frac{1}{2} \sum_{i=1}^{n} \frac{L(T(\gamma_i(s), \gamma_{i+1}(s)))}{(n - 1)d(\gamma_1(s), \gamma_2(s))}$$

$$= \frac{1}{2} \frac{n}{n - 1} \frac{\sum_{i=1}^{n} L(T(\gamma_i(s), \gamma_{i+1}(s)))}{nd(\gamma_1(s), \gamma_2(s))}$$

$$= \frac{1}{2} \frac{n}{n - 1} \frac{\sum_{i=1}^{n} L(T(\gamma_i(s), \gamma_{i+1}(s)))}{\sum_{i=1}^{n} d(\gamma_i(s), \gamma_{i+1}(s))}.$$
Then it follows by (6.13):

\[
\lim_{s \to \infty} \frac{L(\text{SMT for } (N(s)))}{L(\text{MST for } (N(s)))} = \frac{n}{2(n - 1)}.
\] (6.15)

Since this must be true for all integers \( n > 2 \), the proof is complete. \( \square \)
Chapter 7

Related questions

Of course, we may assume, that there are several modifications and relatives of Steiner Problem, and consequently, quantities which are in relatives of the Steiner ratio.

7.1 $k$-SMT’s

We consider the problem of finding a $k$-SMT, which allows at most $k$ Steiner points in the shortest tree.

**Assumption:** There is a positive integer $c = c(X, \rho)$, depending on the space only, such that the degree for any Steiner point in each $k$-SMT for a given set in $(X, \rho)$ is at most $c$. The number $c = c(X, \rho)$ does not depend on the number $k$, that means we can determine $c$ for a 1-SMT.

If $m(X, \rho) = 1$, then any SMT and any $k$-SMT is an MST. Otherwise, if $m(X, \rho)$ is less than one, then $c(X, \rho) \geq 3$.

For the values of the number $c$ for some metric spaces see [21]. Particularly, we saw that for Banach-Minkowski spaces $M_d(B)$ such a value always exists.

A $k$-SMT for a finite set of $n$ points in a metric space which satisfies the assumptions can be found in polynomially bounded time, [21].

Let $k$ and $k'$ be integers with $0 \leq k' \leq k \leq \infty$. We define the restricted Steiner ratio of the metric space $(X, \rho)$ by

$$m(X, \rho)(k : k') = \inf \left\{ \frac{L(k\text{-SMT for } N)}{L(k'\text{-SMT for } N)} : N \text{ is a finite set in } (X, \rho) \right\}.$$  \hspace{1cm} (7.1)

(For $k < k'$ this quantity is undefined.)

**Observation 7.1.1** It holds

$$1 \geq m(X, \rho)(k : k') \geq m(X, \rho) \geq 1/2$$
for any metric space \((X, \rho)\), \(k' \leq k\).

The ratio \(m(k : k - 1)\) is of special interest. To estimate it we will use the local version of Steiner’s Problem, the so-called Fermat’s Problem: Let \(N\) be a finite set of points in \((X, \rho)\). Determine a point in the space such that the function

\[
F_{N}(w) = \sum_{v \in N} \rho(v, w)
\]

(7.2)
is minimal. Each point which minimizes the function \(F_{N}\) is called a Torricelli point for \(N\) in \((X, \rho)\).

**Lemma 7.1.2** Let \(N\) be a finite set of \(n\) points in a metric space. Let \(q\) be a Torricelli point for \(N\) and let \(T_{o}\) be an MST for \(N\). Then

\[
\frac{F_{N}(q)}{L(T_{o})} \geq \frac{n}{2n - 2}.
\]

**Proof.** Let \(N = \{v_1, \ldots, v_n\}\). If \(q\) is in \(N\), then \(F_{N}(q) \geq L(T_{o})\) and the ratio is at least one.

Now, we assume that \(q\) is not in \(N\). Without loss of generality, \(\rho(v_1, v_n)\) is the greatest distance between points of \(N\). Hence,

\[
2(n - 1)F_{N}(q) = (n - 1) \left( \sum_{i=1}^{n} \rho(v_i, q) + \sum_{j=1}^{n} \rho(v_j, q) \right)
\]

\[
\geq (n - 1) \left( \sum_{i=1}^{n-1} \rho(v_i, v_{i+1}) + \rho(v_1, v_n) \right)
\]

\[
\geq (n - 1)L(T_{o}) + (n - 1)\rho(v_1, v_n)
\]

\[
\geq (n - 1)L(T_{o}) + \sum_{i=1}^{n-1} \rho(v_i, v_{i+1})
\]

\[
\geq (n - 1)L(T_{o}) + L(T_{o})
\]

\[
= nL(T_{o}).
\]

---

1Surveys about Fermat’s problem in the form of monographs are given by
6. A.Schöbel: "Locating Lines and Hyperplanes", 1999, [100].
Theorem 7.1.3 In a metric space \((X, \rho)\) which satisfies the assumptions it holds

\[
m(X, \rho)(k : k - 1) \geq \frac{k}{k + 2 - \frac{1}{c(X, \rho)}} > \frac{k}{k + 2}
\]

for all \(k > 0\).

Proof. Let \(T = (V, E)\) be a \(k\)-SMT for \(N\). Then the degree for all Steiner points \(v\) is at most \(c = c(X, \rho)\).
If \(|V| < |N| + k\), then \(T\) also is a \((k - 1)\)-SMT, and the ratio equals one. Now we assume that \(|V| = |N| + k\).
Let \(q \in V \setminus N\), such that the star \(T_s\) induced by \(q\) and its set \(V_s\) of neighbors in \(T\) has minimal length. Let \(T_c\) be an MST for \(V_s\). Clearly, \(L(T_s) \leq L(T_c)\). On the other hand, by the lemma 7.1.2 and the fact that the real function \(x/(2x - 2)\) is monotonically decreasing it follows

\[
L(T_s) \geq \frac{c}{2c - 2} \cdot L(T_c).
\]

\(T'\) is the tree built up by \(T\) with \(T_c\) instead of \(T_s\). Then \(T'\) is a tree with at most \(k - 1\) Steiner points. On the one hand,

\[
L((k - 1)\text{-SMT for } N) \leq L(T') = L(T) - L(T_s) + L(T_c) \leq L(k\text{-SMT for } N) - L(T_s) + (2 - 2/c)L(T_s) = L(k\text{-SMT for } N) + (1 - 2/c)L(T_s).
\]

On the other hand,

\[
L(k\text{-SMT for } N) = L(T) \geq \frac{1}{2} \sum_{v \in V \setminus N} L(\text{star induced by } v \text{ and its neighbors}) \geq \frac{1}{2} \sum_{v \in V \setminus N} L(T_s) = k \cdot \frac{L(T_s)}{2}.
\]

These two inequalities imply the assertion. \(\square\)
The theorem shows that the best addition of $k$ Steiner points to the initial set of given points cannot improve drastically the approximation in comparison to the best addition of $k-1$ Steiner points, if $k$ is a large number. In other terms: The "relative defect" going from a $(k-1)$-SMT to a $k$-SMT for a finite set in a metric space tends to zero, when $k$ runs to infinity.

For instance, we consider the $d$-dimensional affine space with rectilinear distance. Let $N = \{\pm(1,0,\ldots,0)\ldots,\pm(0,\ldots,0,1)\}$, that means the convex hull of $N$ is the unit ball of the space. Clearly, an MST $T$ for $N$ has length $4d - 2$ and the origin is a Torricelli point for $N$. This implies $F_N/L(T) = d/(2d - 1)$. In other words,

$$m(k : k-1) \geq \frac{k}{k + 2 - \frac{2}{d}}$$

for $k \geq 1$. Hence, the inequality in 7.1.3 is the best possible one in the class of all metric spaces.

### 7.2 SMT($\alpha$)

At the end of the former section we saw that the best addition of $k$ Steiner points to the initial set of given points cannot improve drastically the approximation in comparison to the best addition of $k-1$ Steiner points, if $k$ is a large number. More exactly: Let $N$ be a finite set of points in a Banach-Minkowski space. Then the relative defect when going from a $(k-1)$-SMT to a $k$-SMT for $N$ tends to zero, if $k$ runs to infinity. Now, we will use this fact to estimate the number $k$ for $k$-SMT’s depending on the number $\alpha$ for SMT($\alpha$).

Denote by $T_k$ a $k$-SMT for $N$. Then

$$C(T_k) \leq \alpha \cdot k + L(T_k).$$

(7.4)

If $T_k$ contains at most $k-1$ Steiner points, then we know that it is also a $(k-1)$-SMT for $N$ and it holds

$$L(T_k) = L(T_{k-1}).$$

(7.5)

In any case we have (7.1.3) in

$$L(T_{k-1}) \geq L(T_k) \geq \frac{k}{k + \Delta} \cdot L(T_{k-1}),$$

(7.6)

whereby the parameter

$$\Delta = \Delta_d(B) = 2 - \frac{4}{c_d(B)}$$

is a positive real number, namely

$$\Delta \geq \frac{2}{3},$$

(7.7)
since we have \( c \geq 3 \).

Now we consider the costs of the \( k \)-SMT’s. If \( T_{k-1} = T_k \), which means that a \( k \)-SMT uses at most \( k - 1 \) Steiner points, then we have \( C(T_{k-1}) = C(T_k) \). In the other case we find

\[
C(T_k) = \alpha \cdot k + L(T_k). 
\]  
(7.8)

We are interested in the condition

\[
C(T_k) \leq C(T_{k-1}). 
\]  
(7.9)

Recalling (7.4) and (7.8) we see that this condition is equivalent to

\[
\alpha \cdot k + L(T_k) \leq \alpha \cdot (k - 1) + L(T_{k-1}). 
\]  
(7.10)

Hence, we get that the insertion of a new Steiner point is only sensible if the difference between the lengths of trees is at least the value of the parameter \( \alpha \):

\[
\alpha \leq L(T_{k-1}) - L(T_k). 
\]  
(7.11)

Furthermore, in view of (7.6) we have

\[
L(T_{k-1}) - L(T_k) \leq \frac{\Delta}{k} L(T_k) 
\]  
(7.12)

Both inequalities (7.11) and (7.12) imply

\[
\alpha \leq \frac{\Delta}{k} L(T_k). 
\]  
(7.13)

In other terms, the insertion of a new Steiner point is only sensible if (7.13) holds. Conversely,

**Theorem 7.2.1** If we are looking for an SMT(\( \alpha \)) for a set \( N \) of given points in a Banach-Minkowski space \( M_d(B), d \geq 2 \), we are only interested in the \( k \)-SMT’s for \( N \) with

\[
k \leq \frac{\Delta}{\alpha} \cdot L(MST \ for \ N),
\]

where

\[
\Delta = 2 - \frac{4}{cd(B)}.
\]

### 7.3 Greedy Trees

In 1992, Smith and Shor [104] introduced the notion of a so-called Greedy Tree (GT) for a set \( N \) of points in a Euclidean space as follows:

1. Start with all points of \( N \), regarded as a forest of \( n = |N| \) single vertices;
2. At any stage, add the shortest possible segment to the current forest, which causes two trees to merge;

3. Continue until the forest is completely merged into one tree.

Greedy Trees are simple to construct and have the following properties:

**Observation 7.3.1** (Smith and Shor [104]) Let $T = (V, E)$ be a GT for $N$ in a Euclidean space. Then it holds

(a) $T$ is an MST for $V$.

(b) Any edge $e \in E$ which connects two points of $N$ is also an edge of a (desired chosen) MST for $N$.

(c) The GT $T$ is no longer than an MST for $N$. Hence,

$$\frac{L(SMT \text{ for } N)}{L(T)} \geq \frac{L(SMT \text{ for } N)}{L(MST \text{ for } N)} \geq m,$$

where $m$ denotes the Steiner ratio of the space.

It is conjectured that the ratio between an SMT and a GT is greater than the Steiner ratio of the space. More exactly:

**Conjecture 7.3.2** (Smith and Shor [104])

$$\inf \left\{ \frac{L(SMT \text{ for } N)}{L(GT \text{ for } N)} : N \subseteq \mathbb{L}_2^2 \text{ is a finite set} \right\} = \frac{2\sqrt{3}}{2 + \sqrt{3}} = 0.9282 \ldots$$

This bound is achieved by the three points of an equilateral triangle.

In high dimensions the advantage of GT’s over MST’s can become quite pronounced: Let $N(d)$ be the the nodes of a regular simplex of unit side length in the $d$-dimensional Euclidean space. Then an MST has length $= d$, and an GT for $N(d)$ has length

$$\sum_{1 \leq k \leq d} \sqrt{\frac{k + 1}{2k}} \sim 0.7071 \cdot d \quad (7.14)$$

for $d \to \infty$. On the other hand, we have an upper bound for the Steiner ratio of $0.66984 \cdot d$, see 3.11.2.

### 7.4 Component-size bounded Steiner Trees

There is an approximation method for Steiner’s Problem which uses trees that can contain Steiner points, but not in an arbitrary sense: Let $N$ be a finite set of points in a metric space $(X, \rho)$. Let $T = (V, E)$ be a tree interconnecting $N$. For such trees we assume that the degree of each given point is at least one and the degree of each
Steiner point in $V \setminus N$ is at least three. However, a given point in such a tree may not be a leaf. When a given point $v$ is not a leaf, $T$ can be decomposed (by splitting at the given point) into several smaller trees, so that given points only occur as leaves. More precisely:

1. Define $G = (V \setminus \{v\}, E \setminus \{vv' : v'$ is a neighbor of $v\})$. ($G$ is a forest with $g(v)$ components $G_i = (V_i, E_i)$, $i = 1, \ldots, g(v)$.)

2. Define for $i = 1, \ldots, g(v)$ the graph $G_i = (V_i \cup \{v_i\}, E_i \cup \{v_iv' : v'$ is a neighbor of $v$ in $G$ and $v'$ is in $V_i\})$, where $v_i$ is not in $V$.

In this way, every tree interconnecting $N$ is decomposed into so-called full components. The size of a full component is the number of given points in the full component.

A $k$-size tree for $N$ is a tree interconnecting all points of $N$ with all full components of size at most $k$. A $k$-size SMT is the shortest one among all $k$-size trees. For $k = 2$ we look for an MST. For every $k \geq 4$ this problem is NP-hard, [96].

Clearly, we are interested in the greatest lower bound for the ratio between the lengths of an SMT and a $k$-size SMT for the same set of points in a metric space:

$$m^{(k)}(X, \rho) = \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{k-size SMT for } N)} : N \subseteq (X, \rho) \text{ is a finite set} \right\}.$$  
(7.15)

This quantity is called the $k$-size-Steiner ratio of the metric space $(X, \rho)$.

In any metric space $(X, \rho)$ an $2$-size SMT is an MST. Hence, the $2$-size-Steiner ratio is the Steiner ratio:

$$m^{(2)}(X, \rho) = m(X, \rho).$$  
(7.16)

Furthermore, 

**Observation 7.4.1** For the $k$-size-Steiner ratio $m^{(k)}$, $k > 2$ the following is known:

(a) (Zelikovsky [120]) For any metric space $(X, \rho)$ it holds that

$$m^{(3)}(X, \rho) \geq \frac{3}{5} = 0.6.$$  
(7.17)

(Du [45]) This lower bound is the best possible one over the class of all metric spaces.

(b) (Du [43]) For any metric space $(X, \rho)$ it holds that

$$m^{(k)}(X, \rho) \geq \frac{r}{r + 1},$$  
(7.18)

where $r = \lfloor \log_2 k \rfloor$. 80
Now we can describe the performance ratio of approximations for Steiner’s Problem more exactly. Zelikovsky [120] showed that there exists a polynomial-time approximation \( A \) for Steiner’s Problem in a metric space \((X, \rho)\) with performance ratio

\[
\text{error}(A) = \frac{1}{2} \left\{ \frac{1}{m^{(3)}(X, \rho)} + \frac{1}{m^{(2)}(X, \rho)} \right\},
\]  

provided that an SMT for three given points can be computed in polynomial time. Using a similar idea, Berman and Ramaiyer [8] showed that there is a polynomial-time approximation \( A_k \) with performance ratio

\[
\text{error}(A_k) \geq \frac{1}{1 \cdot 2} \cdot \frac{1}{m^{(2)}(X, \rho)} + \frac{2}{2 \cdot 3} \cdot \frac{1}{m^{(3)}(X, \rho)} + \frac{1}{3 \cdot 4} \cdot \frac{1}{m^{(4)}(X, \rho)} + \ldots,
\]  

provided that for any \( k \) an SMT for \( k \) points can be computed in polynomial time.

Clearly, we are interested in the \( k \)-size-Steiner ratio for specific spaces. For the plane with rectilinear distance we have

<table>
<thead>
<tr>
<th>( k )</th>
<th>( m^{(k)} = ) Source</th>
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<tbody>
<tr>
<td>2</td>
<td>( \frac{2}{3} ) Hwang, [66]</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{3}{5} ) Berman and Ramaiyer, [8]</td>
</tr>
<tr>
<td>( \leq 4 )</td>
<td>( \frac{2k-1}{2k} ) Borchers et al., [11].</td>
</tr>
</tbody>
</table>

Such nice results for the Euclidean plane are not yet known. Borchers and Du [11] determine the \( k \)-size-Steiner ratio for graphs exactly: For \( k = 2^r + s \), where \( 0 \leq s < 2^r \), this quantity is

\[
m^{(k)}(G) = \frac{r \cdot 2^r + s}{(r + 1) \cdot 2^r + s}.
\]  

7.5 Steiner’s Problem in spaces with a weaker triangle inequality

Up to now, we have used the triangle inequality as a property of the metric. It is conceivable that slight violations of the triangle inequality should not be too deleterious with respect to performance guarantees of an approximation. Andreae and Bandelt [5] consider the deviation from the triangle inequality captured by a parameter \( \tau \) in the following relaxation:

\[
\rho(v, v') \leq \tau(\rho(v, w) + \rho(w, v'))
\]  

for all \( v, v', w \in X \).

Such a parametrized triangle inequality is given in the situation that the input data are from a fixed range of values. Assume that all distances under consideration are bounded by real numbers \( L \) and \( U \) in the following way:

\[
L \leq \rho(v, v') \leq U
\]  

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for different points \( v \) and \( v' \).

For instance, for a network \( G \) we have \( L = 1 \) and \( U = \text{diam} G \).

If \( L > 0 \) then \( \rho(v, w) + \rho(w, v') \geq 2L \), so that \( U(\rho(v, w) + \rho(w, v')) \geq 2L\rho(v, v') \).

Hence,

**Observation 7.5.1** The metric \( \rho \) satisfies the inequality (7.23) with the parameter

\[
\tau = \frac{U}{2L} \geq \frac{1}{2}.
\]  

This scenario applies to the minimum spanning tree approximation for Steiner’s Problem: When the parameter \( \tau \) approaches \( 1/2 \), the performance guarantee factor 2 decreases and eventually reaches 1; recall 2.2.1. We can see that the factor decreases when we make the additional assumption that, for some \( \tau \) with \( 0 < \tau \leq 1 \), the set \( N \) of given points satisfies the following inequality:

\[
\rho(v, v') \leq \tau \cdot (\rho(v, w) + \rho(w, v')) \quad (7.25)
\]

for all \( v, v' \in N \) and \( w \in X \setminus N \). Then the following is true:

**Theorem 7.5.2** (Andreae, Bandelt [5]) Let \( (X, \rho) \) be a metric space, and let \( N \) be a finite subset of \( X \) with \( |N| = n > 1 \). Let \( 0 < \tau \leq 1 \). Suppose that \( N \) satisfies equation (7.25) with respect to \( \tau \).

Let \( T \) be an SMT and \( T' \) be an MST for \( N \) in \( (X, \rho) \). Then

\[
L(T') \leq 2 \cdot \tau \cdot \left(1 - \frac{1}{n}\right) \cdot L(T)
\]

if \( \tau \geq n/(2n - 2) \), and

\[
L(T') = L(T)
\]

otherwise.

The following example shows that the bound given in 7.5.2 is the best possible:

Consider \( X = N \cup \{x\} \) with the distances \( \rho(v, v') = 2\tau \) for different points \( v \) and \( v' \), and \( \rho(v, x) = 1 \).

### 7.6 The average case

The Steiner ratio is a quantity to describe a worst-case scenario. On the other hand, the average-case is also of interest. More exactly: Distribute \( n \) points \( v_1, \ldots, v_n \) by a suitable random process in the space \( (X, \rho) \) and then ask for the expected value

\[
E(n) = E(X, \rho)(n) \text{ of } \frac{L(\text{SMT for } \{v_1, \ldots, v_n\})}{L(\text{MST for } \{v_1, \ldots, v_n\})}.
\]
Very little is known about these functions. Clearly,

\[ E(X, \rho)(n) \geq m^n(X, \rho) \geq m(X, \rho), \quad (7.26) \]

where

\[ m^n(X, \rho) := \inf \left\{ \frac{L(X, \rho)(SMT \text{ for } N)}{L(X, \rho)(MST \text{ for } N)} : N \subseteq X, |N| \leq n \right\}. \quad (7.27) \]

Values of \( E(n) = E(X, \rho)(n) \) for specific spaces and distributions of points are given by [59], [67] and [115].
Chapter 8

Summary

Steiner’s Problem is very hard as well in combinatorial as in computational sense, but, on the other hand, the determination of an MST is simple. Consequently, we are interested in the greatest lower bound for the ratio between the lengths of these trees, which is called the Steiner ratio (of the space $X$):

$$m(X) := \inf \left\{ \frac{\text{length}(\text{SMT for } N)}{\text{length}(\text{MST for } N)} : N \text{ a finite set in the space } X \right\}.$$ 

It is not hard to see, by Moore, that for Steiner ratio of every metric space $1 \geq m(X, \rho) \geq \frac{1}{2}$ holds. Ivanov and Tuzhilin showed that for any real number between 0.5 and 1 there is a metric space with this Steiner ratio. This remains true, when we restrict ourself to finite spaces.

The exact value for the Steiner ratio is only known for very few metric spaces:

- Ultrametric spaces have Steiner ratio 1.
- In the class of all finite-dimensional Banach spaces, a space has Steiner ratio 1 if and only if the dimension equals 1.
- The plane with rectilinear norm has Steiner ratio $2/3$.
- Phylogenetic spaces have Steiner ratio $1/2$.
- The space of all convergent sequences with supremum norm has Steiner ratio $1/2$.
- The Steiner ratio of a simply connected manifold of negative constant curvature without boundary equals $1/2$. 

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We know the Steiner ratio of some finite metric spaces.

An interesting problem, but which seems as very difficult, is to determine the range of the Steiner ratio for $d$-dimensional Banach spaces, depending on the value $d$. More exactly, determine the best possible reals $c_d$ and $C_d$ such that

$$c_d \leq m(X) \leq C_d,$$

where $X$ is a Banach space of dimension $d$.

Two conjectures: For $d = 2, 3, \ldots$

$$C_d = m(d, 2),$$

where $m(d, 2)$ denotes the Steiner ratio of the $d$-dimensional Euclidean space. And

$$c_d > 1/2.$$

For several metric spaces it is shown that they have the same Steiner ratio as the Euclidean spaces:

- The Steiner ratio for spheres.
- The Steiner ratio for a flat two-dimensional torus, a flat Klein bottle, a projective plane having constant positive curvature.
- The Steiner ratio of Einstein-Riemann spaces.

All these results show the importance of the knowledge of $m(d, 2)$. But the exact values of these quantities are not known. This is also true for the Euclidean plane, where we have the well-known conjecture by Gilbert and Pollak:

$$m(2, 2) = \frac{\sqrt{3}}{2} = 0.86602 \ldots.$$

When the dimension go to infinity, the Steiner ratio decreases:

$$1 = m(1, 2) \geq m(2, 2) \geq m(3, 2) \geq m(4, 2) \geq \ldots \geq \lim_{d \to \infty} m(d, 2) \geq 0.615 \ldots,$$

whereby the last inequality was shown by Du.

More generally, we are interested in the sequence

$$1 = m(1, p) \geq m(2, p) \geq m(3, p) \geq m(4, p) \geq \ldots \geq \lim_{d \to \infty} m(d, p) \geq 0.5.$$

To determine these values is a non-trivial question, since it is well-known that several facts useful to attack Steiner’s Problem are only true in Banach-Minkowski planes but they are not true in Banach-Minkowski spaces of higher dimensions.

An interesting problem is to determine the range of the Steiner ratio of infinite-dimensional Banach spaces: The lower bound is easy to compute by $1/2$, the upper bound $C_\infty$ is still open.
Bibliography


